WEAKLY 2-RANDOMS AND 1-GENERICS IN SCOTT SETS

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ABSTRACT. Let $\mathcal{S}$ be a Scott set, or even an ω-model of WWKL. Then for each $A \in \mathcal{S}$, either there is $X \in \mathcal{S}$ that is weakly 2-random relative to A, or there is $X \in \mathcal{S}$ that is 1-generic relative to A. It follows that if $A_1, \dots, A_n \in \mathcal{S}$ are non-computable, there is $X \in \mathcal{S}$ such that each A_i is Turing incomparable with X, answering a question of Kučera and Slaman. More generally, any $\forall\exists$ sentence in the language of partial orders that holds in $\mathcal{D}$ also holds in $\mathcal{D}^{\mathcal{S}}$, where $\mathcal{D}^{\mathcal{S}}$ is the partial order of Turing degrees of elements of $\mathcal{S}$.

Kučera and Slaman [2] showed that for every Scott set $\mathcal{S}$ and every non-computable $A \in \mathcal{S}$, there is $X \in \mathcal{S}$ such that X and A are Turing incomparable. Their non-uniform proof hinged on whether A was a K-trivial set. Conidis [1] generalized this result to require only that $\mathcal{S}$ is an ω-model of WWKL (meaning that for every $A \in \mathcal{S}$, every $\Pi^0_1(A)$ class *of positive measure* has an element in $\mathcal{S}$). Kučera and Slaman left open whether, given non-computable $A, B \in \mathcal{S}$, one could find $X \in \mathcal{S}$ incomparable with both. We answer that question in the affirmative, generalizing both of the above results (Corollary 1).

This is related to the question, asked by Kučera and Slaman, of which $\forall\exists$ sentences (in the language of partial orders) hold in $\mathcal{D}^{\mathcal{S}}$, the partial order induced by $\leq_T$ on the degrees of $\mathcal{S}$. One direction of the proof of the decidability of the $\forall\exists$ theory of the Turing degrees $\mathcal{D}$ rests on a technical result (cf. [3, Theorem II.4.11]) which we generalize from 2^ω to an arbitrary Scott set $\mathcal{S}$ in Theorem 3. Whenever Theorem 3 holds for some $\mathcal{S}$, then every $\forall\exists$ sentence that holds in $\mathcal{D}$ also holds in $\mathcal{D}^{\mathcal{S}}$ (Corollary 2). Li and Slaman [4] recently showed that whenever $\mathcal{S} \subseteq \Delta^0_2$ is a Scott set and $A \in \mathcal{S}$, then there is $G \in \mathcal{S}$ such that G is 1-generic relative to A. Theorem 3 and Corollary 2 for the case where $\mathcal{S} \subseteq \Delta^0_2$ follow immediately.

Theorem 2 is the new result which allows the usual proof of Theorem 3 for $\mathcal{S} = 2^\omega$ to lift to arbitrary ω-models of WWKL. Corollary 1 is also a special case of Theorem 3.

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To prove Theorem 2, we relativize the following theorem about weakly 2-random ($W2R$) sets.

Theorem 1 (Hirschfeldt-Miller, cf. [5, Thm 5.3.16])**.** *An ML-random set is weakly 2-random if and only if it does not compute any non-computable c.e. set.*

Theorem 2. *For any Scott set $\mathcal{S}$ and any $A \in \mathcal{S}$, either there is $X \in \mathcal{S}$ that is weakly 2-random relative to A, or there is $X \in \mathcal{S}$ that is 1-generic relative to A.*

Proof. Let $\mathcal{P}$ be a $\Pi^0_1(A)$ class consisting entirely of MLR^A reals. Let $X \in \mathcal{S} \cap \mathcal{P}$, and suppose that X is not $W2R^A$. By the relativization of Theorem 1 to A, there is a set $D \leq_T X \oplus A$ such that D is c.e. in A and $D \not\leq_T A$. Relativizing the proof that every non-computable c.e. set computes a 1-generic, there is a $G \leq_T D \oplus A$ such that G is 1-generic relative to A. □

Because $\mathcal{P}$ has positive measure, Theorem 2 and all the other results here hold even if $\mathcal{S}$ is only an ω-model of WWKL. If every element of $\mathcal{S}$ is Δ^0_2, then no element is $W2R$, and if every element of $\mathcal{S}$ is hyperimmune-free, then no element is 1-generic, so the result is sharp.

Corollary 1. *For any Scott set $\mathcal{S}$ and any non-computable $A_1, \dots, A_n \in \mathcal{S}$, there is $X \in \mathcal{S}$ such that X is Turing incomparable with each A_i.*

Proof. Let $A = \oplus_{i\leq n} A_i$. If some $X \in \mathcal{S}$ is 1-generic relative to A, then X is 1-generic relative to each A_i, which suffices. So suppose some $X \in \mathcal{S}$ is $W2R^A$. Then for each i, $X \not\geq_T A_i$, because $\{Y : Y \geq_T A_i\}$ is a $\Sigma^0_3(A_i)$ set of measure 0. □

The same method gives a more general extension of embeddings result, entirely analogous to the extension of embeddings result that holds in $\mathcal{D}$. The only addition is the observation that a weakly 2-random can play the same role as a 1-generic in the usual proof of this theorem in $\mathcal{D}$.

If $\mathcal{H} = (H, \leq_H), \mathcal{M} = (M, \leq_M)$ are partial orders, then $\mathcal{H} \prec \mathcal{M}$ means that $H \subseteq M$ and $\leq_H$ and $\leq_M$ agree on elements of H. We say that $\mathcal{M}$ is an *end-extension* of $\mathcal{H}$ if for every $h \in H$ and $m \in M \setminus H$, we have $m \not\leq_M h$. If $\mathcal{H}$ is an upper semi-lattice with join $\vee_H$, we say that $\mathcal{M}$ *respects the joins of* $\mathcal{H}$ if for every $h, k \in H$ and $m \in M \setminus H$, if $m \geq_M h$ and $m \geq_M k$, then $m \geq_M h \vee_H k$.

Theorem 3. *Let $\mathcal{H} \prec \mathcal{M}$ be finite partial orders, where $\mathcal{H}$ is an upper semi-lattice. Let $\mathcal{S}$ be a Scott set, and let $f : H \to \mathcal{S}$ be an upper semi-lattice embedding of $\mathcal{H}$ into $(\mathcal{S}, \leq_T, \oplus)$. Suppose $\mathcal{M}$ is an end-extension of $\mathcal{H}$ which respects the joins of $\mathcal{H}$. Then f can be extended to $\hat{f} : M \to \mathcal{S}$ which is a partial order embedding of $\mathcal{M}$ into $(\mathcal{S}, \leq_T)$.*

Proof. For $h \in H$, let $A_h = f(h)$. Let $G \in \mathcal{S}$ be either 1-generic or weakly 2-random relative to $\bigoplus_{h\in H} A_h$. Interpret G as the join of finitely many columns, one for each $m \in M \setminus H$, so that the reals G_m are defined by $G = \bigoplus_{m\in M\setminus H} G_m$. For each $m \in M \setminus H$, define

$$A_m = \bigoplus_{\substack{h\in H \\ h\leq_M m}} A_h \oplus \bigoplus_{\substack{\ell\in M\setminus H \\ \ell\leq_M m}} G_\ell.$$

We claim that $\hat{f}(m) := A_m$ satisfies the conclusion of the theorem. If G is 1-generic, we have just repeated the usual construction. If G is weakly 2-random, there is only one verification step that is different: we claim that if $h \in H$ and $m \in M \setminus H$ with $h \not\leq_M m$, then $A_h \not\leq A_m$. Let

$$\mathcal{B} = \{Z : \bigoplus_{\substack{k\in H \\ k\leq_M m}} A_k \oplus Z \geq_T A_h\}$$

This set is $\Sigma^0_3(\bigoplus_{k\in H} A_k)$. If it had positive measure, a majority vote argument would provide a computation to show $\bigoplus_{\substack{k\in H \\ k\leq_M m}} A_k \geq_T A_h$. This is not possible because $m \not\geq_M h$ and $\mathcal{M}$ respects the joins of $\mathcal{H}$. Therefore $\mathcal{B}$ is null, so $G \notin \mathcal{B}$, so $A_m \notin \mathcal{B}$.

All the other steps in the verification are the same after the observation that the relative ML-randomness of G guarantees that $\bigoplus_{h\in H} A_h \oplus \bigoplus_{\ell\neq m} G_\ell \not\geq_T G_m$ for all $m \in M \setminus H$. □

We can now give a partial answer to the question asked in [2] of whether the $\forall\exists$ theory (in the language of partial orders) of the degrees of a Scott set coincides with the $\forall\exists$ theory of $\mathcal{D}$.

Corollary 2. *For any Scott set $\mathcal{S}$, any $\forall\exists$ sentence that holds in $\mathcal{D}$ also holds in $\mathcal{D}^{\mathcal{S}}$.*

Proof. An *extension sentence* is a sentence of the form $\forall\bar{x}\exists\bar{y}(\theta(\bar{x}) \to \bigvee_{j<n} \eta_j(\bar{x}, \bar{y}))$, where $\theta(\bar{x})$ and each $\eta_j(\bar{x}, \bar{y})$ are conjunctions of atomic formulas and negated atomic formulas describing the complete $\leq$-diagram on their inputs, where the diagram described by θ is an upper semi-lattice, and where each η_j describes a diagram extending the diagram given by θ. It is known (cf. [3, Theorem VII.4.4]) that for any $\forall\exists$ sentence ψ, there is a finite conjunction of extension sentences ϕ, such that in any upper semi-lattice, ψ holds if and only if all the conjuncts hold. Furthermore, in $\mathcal{D}$, an extension sentence ϕ holds if and only if there is a j such that the diagram $\mathcal{M}_j$ described by η_j is an end extension of the diagram $\mathcal{H}$ described by θ, and $\mathcal{M}$ respects the joins of $\mathcal{H}$ (cf. [3, Theorems II.4.11, VII.4.1]).

Applying Theorem 3, if an extension sentence ϕ holds in $\mathcal{D}$, then ϕ also holds in $\mathcal{D}^{\mathcal{S}}$. Therefore, if ψ holds in $\mathcal{D}$, then all the conjuncts ϕ hold in both structures, so ψ holds in $\mathcal{D}^{\mathcal{S}}$ as well. □

It remains open whether all Scott sets satisfy exactly these $\forall\exists$ sentences.

References

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