Turing, tt-, and m-reductions for functions in the Baire hierarchy

Linda Brown Westrick
University of Connecticut
Joint with Adam Day and Rod Downey

July 27, 2017
Computability & Complexity in Analysis
Daejeon

Computable reducibility for discontinuous functions

Motivating question: Suppose $f, g : 2^\omega \to \mathbb{R}$.

(maybe f and g are very discontinuous)

What should $f \leq_T g$ mean?

Computable reducibility for discontinuous functions

Motivating question: Suppose $f, g : 2^\omega \to \mathbb{R}$.

(maybe f and g are very discontinuous)

What should $f \leq_T g$ mean?

Some intuition:

- Shifting or scaling a function by a computable factor should not change the difficulty of computing it.
- Given f, g, their join $f \oplus g$ should have the same degree as a function consisting of a scaled copy of f next to a scaled copy of g.
- Given f, g, we should have $f + g \leq_T f \oplus g$.
- A step function that steps at some $X \in 2^\omega$ should compute a step function that steps at any $Y \leq_T X$.

Continuous strong parallelized Weihrauch reducibility

Motivating question: Suppose $f, g : 2^\omega \to \mathbb{R}$.

(maybe f and g are very discontinuous)

What should $f \leq_T g$ mean?

Definition. Say that $f \leq_{\mathbf{T}} g$ if and only if $f \leq_{sW}^{c} \hat{g}$.

That is, $f \leq_{\mathbf{T}} g$ if and only if there are continuous functions $h_0, h_1, \dots$ and k such that for all $X \in 2^\omega$, whenever Y_i are names for $g(h_i(X))$, then $k(\oplus_i Y_i)$ is a name for $f(X)$.

Examples:

- For any g and any computable $Y \in 2^\omega$, if $f(X) = g(X + Y)$, where addition is componentwise mod 1, then $f \leq_{\mathbf{T}} g$.

Examples

Definition. Say that $f \leq_{\mathbf{T}} g$ if and only if $f \leq_{sW}^{c} \hat{g}$.

That is, $f \leq_{\mathbf{T}} g$ if and only if there are continuous functions $h_0, h_1, \ldots$ and k such that for all $X \in 2^\omega$, whenever Y_i are names for $g(h_i(X))$, then $k(\oplus_i Y_i)$ is a name for $f(X)$.

Examples:

- For any f and g, we have $f + g \leq_{\mathbf{T}} t$, where

$$t(i^\frown X) = \begin{cases} f(X) & \text{if } i = 0 \\ g(X) & \text{if } i = 1 \end{cases}$$

- For $Z \in 2^\omega$, let s_Z be a step function that steps at Z.

$$s_Z(X) = \begin{cases} 0 & \text{if } X \leq_{lex} Z \\ 1 & \text{if } X >_{lex} Z. \end{cases}$$

Then $s_{0^\omega} \leq_{\mathbf{T}} s_{(01)^\omega}$.

Examples

Definition. Say that $f \leq_{\mathbf{T}} g$ if and only if $f \leq_{sW}^{c} \hat{g}$.

That is, $f \leq_{\mathbf{T}} g$ if and only if there are continuous functions $h_0, h_1, \dots$ and k such that for all $X \in 2^\omega$, whenever Y_i are names for $g(h_i(X))$, then $k(\oplus_i Y_i)$ is a name for $f(X)$.

Examples:

- In fact, whenever s_Z is discontinuous, we have $s_Y \leq_{\mathbf{T}} s_Z$ for all $Y \in 2^\omega$.
- If f is continuous and g is non-constant, then $f \leq_{\mathbf{T}} g$.

Baire functions

Recall the Baire hierarchy of functions:

- $\mathcal{B}_0$ is the continuous functions
- $\mathcal{B}_\alpha$ is the set of pointwise limits of functions from $\cup_{\beta<\alpha}\mathcal{B}_\beta$.

For example $s_{0^\omega} \in \mathcal{B}_1 \setminus \mathcal{B}_0$.

Useful equivalent definition:

We have $f \in \mathcal{B}_n$ if and only if there is a computable functional Γ and a parameter $Z \in 2^\omega$ such that for all X,

$$f(X) = \Gamma((X \oplus Z)^{(n)}).$$

At level ω, one jump is "skipped".

$$f \in \mathcal{B}_\omega \iff \text{for some } \Gamma \text{ and } Z\text{, we have } f(X) = \Gamma((X \oplus Z)^{(\omega+1)}).$$

Properties of $\leq_{\mathbf{T}}$

Proposition When restricted to functions from the Baire hierarchy (or, assuming $AD+$, without restriction), the $\equiv_{\mathbf{T}}$ degrees are linearly ordered. Furthermore, within the Baire hierarchy, the degrees are exactly

- The proper Baire classes $\mathcal{B}_{\alpha+1} \setminus B_\alpha$, and
- For each limit ordinal λ, there are two degrees whose union is $\mathcal{B}_\lambda \setminus \cup_{\beta<\lambda} B_\beta$.

Theorem (Kihara). Assume $AD+$. The following degree structures are isomorphic (both are long well-orders):

- The uniformly Turing order preserving jump operators under Martin reducibility
- The discontinuous functions $f : 2^\omega \to \mathbb{R}$ under $\leq_{\mathbf{T}}$

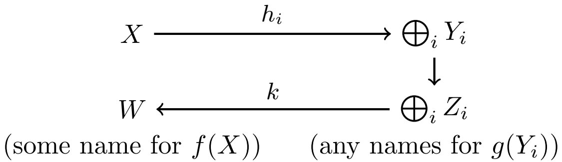
Furthermore, this isomorphism is essentially the identity map.

I won't define those terms, but the map $X \mapsto (X \oplus Z)^{(n)}$ is an example of a uniformly Turing order preserving jump operator.

Truth-table and many-one reducibility

The spirits of tt- and m-reducibility are:

- Truth-table: Say in advance exactly what bits of the oracle you will use, and what you will do with them.
- Many-one: Specify in advance exactly one bit of the oracle, and use its answer as your answer.



Idea: Make k a tt-reduction or an m-reduction.

Problem: What is one bit of information about a real? Cauchy name representation of a real doesn't make much sense for this.

One bit of information

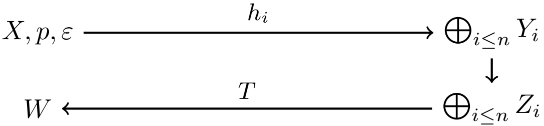
A bit of information about a real number x should be roughly: for a given rational p, say whether $x < p$ or $x > p$.

This is too sharp, so fuzz it up with a rational ε: Given (p, ε), an *acceptable* (p, ε)-bit of x is

$$\begin{cases} 0 & \text{if } x \leq p - \varepsilon \\ 1 & \text{if } x \geq p + \varepsilon \\ 0 \text{ or } 1 & \text{if } p - \varepsilon < x < p + \varepsilon \end{cases}$$

Definition 2. We say $X \in 2^\omega$ is an *acceptable name* for $x \in \mathbb{R}$ if for all $p, \varepsilon \in \mathbb{Q}$, with $\varepsilon > 0$, we have $X(\langle p, \varepsilon \rangle)$ is an acceptable (p, ε)-bit of x.

Definition of *tt*-reducibility



(some acceptable bit for $f(X), p, \varepsilon$) (any acceptable names for $g(Y_i)$)

Definition 3. We say $f \le_{\mathbf{tt}} g$ if for every (p, ε), there are

- continuous functions $h_0, \ldots h_{n-1} : 2^\omega \to 2^\omega$,
- rational pairs $(r_0, \varepsilon_0), \ldots, (r_{n-1}, \varepsilon_{n-1})$, and
- a truth table $T : \{0,1\}^n \to \{0,1\}$

such that whenever b_i are acceptable (r_i, ε_i) bits for $g(h_i(X))$, then $T(b_0, \ldots b_{n-1})$ is an acceptable (p, ε) bit for $f(X)$.

Example: If $f, g : 2^\omega \to \mathbb{R}$ are **bounded** functions, then $f + g \le_{\mathbf{tt}} t$, where

$$t(i^\frown X) = \begin{cases} f(X) & \text{if } i = 0 \\ g(X) & \text{if } i = 1 \end{cases}$$

An equivalent $\leq_{\mathtt{tt}}$ definition

Proposition (Pauly). For $f, g : 2^\omega \to \mathbb{R}$, we have $f \leq_{\mathtt{tt}} g$ if and only if $S_f \leq_{sW}^{c} S_g^*$, where S_f is the Weihrauch Problem "given $(p, \varepsilon), X$, output a (p, ε)-acceptable bit for $f(X)$."

(one direction does use the compactness of 2^ω)

Structure of Baire 1 functions

The Baire 1 functions support several ω_1-length ranking functions.

Consider the α, β and γ ranks studied by Kechris-Louveau (1990), corresponding to three different characterizations of the Baire 1 functions.

The α rank is defined as follows. Given $f \in \mathcal{B}_1$ and $p, \varepsilon \in \mathbb{Q}$, let

- $P^0 = 2^\omega$,
- $P^{\nu+1} = P^\nu \setminus \cup\{U \text{ open } : f(U \cap P) \subseteq (p-\varepsilon, \infty) \text{ or } f(U \cap P) \subseteq (-\infty, p+\varepsilon)\}$
- $P^\nu = \cap_{\mu<\nu} P^\mu$ for ν a limit.

Let $\alpha(f, p, \varepsilon)$ be the least α such that $P^\alpha = \emptyset$.
Let $\alpha(f) = \sup_{p,\varepsilon\in\mathbb{Q}} \alpha(f, p, \varepsilon)$.

The different ranks do not coincide generally, but:

Theorem. (Kechris, Louveau) If $f : 2^\omega \to \mathbb{R}$ is bounded, then for each ordinal ξ, we have

$$\alpha(f) \leq \omega^\xi \iff \beta(f) \leq \omega^\xi \iff \gamma(f) \leq \omega^\xi.$$

Characterization of the $\leq_{\mathtt{tt}}$ degrees in $\mathcal{B}_1$

For $f : 2^\omega \to \mathbb{R}$, let $\xi(f)$ be the least ξ such that $\alpha(f) \leq \omega^\xi$.

Theorem. (DDW) For $f, g \in \mathcal{B}_1$, we have

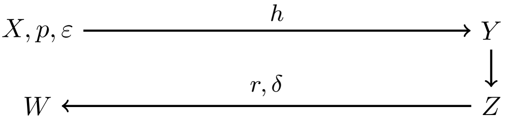
$$f \leq_{\mathtt{tt}} g \iff \xi(f) \leq \xi(g).$$

Corollary. (Kechris-Louveau) If $f, g \in \mathcal{B}_1$ are bounded, then

$$\xi(f+g) \leq \max(\xi(f), \xi(g)).$$

Proof: Observe that (using boundedness) $f + g \leq_{\mathtt{tt}} f \oplus g$.

Definition of m-reducibility



(some acceptable bit for $f(X), p, \varepsilon$) (any acceptable name for $g(Y)$)

Definition 4. We say $f \leq_{\mathbf{m}} g$ if for every (p, ε), there is

- a continuous function $h : 2^\omega \to 2^\omega$, and
- a rational pair (r, δ)

such that whenever b is an acceptable (r, δ) bit for $g(h(X))$, then b is also an acceptable (p, ε) bit for $f(X)$.

Example:

If discontinuous functions s and t are both lower semi-continuous step functions, then $s \equiv_{\mathbf{m}} t$. But if one is lower semi-continuous and the other upper semicontinuous, then they are $\leq_{\mathbf{m}}$-incomparable.

Landmarks in the Baire hierarchy

Definition. Let $j_n : 2^\omega \to \mathbb{R}$ be defined by

$$j_n(X) = \sum_{i \in \omega} \frac{X^{(n)}(i)}{2^{i+1}}.$$

Fact. For each n, we have $j_n \in \mathcal{B}_n$.

Theorem. (DDW)

1. The $\leq_\mathbf{m}$ equivalence classes are almost linearly ordered, and for each $f \in \mathcal{B}_n$, we have $f \leq_\mathbf{m} j_{n+1}$.
2. For each n and f, if f is Baire but $f \notin \mathcal{B}_n$, then either

$$j_{n+1} \leq_\mathbf{m} f \text{ or } -j_{n+1} \leq_\mathbf{m} f.$$

Proof:

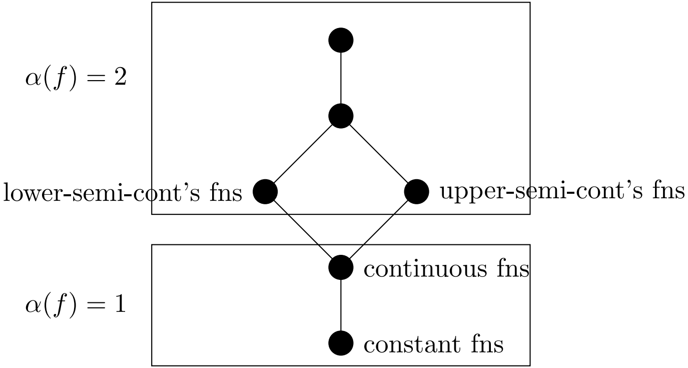
1. A game.
2. Uses $0^{(n)}$ priority argument.

Characterization of the $\leq_{\mathbf{m}}$-degrees in $\mathcal{B}_1$

Theorem. (DDW)

- If $\alpha(f) < \alpha(g)$, then $f <_{\mathbf{m}} g$.
- If $\alpha(f) = \alpha(g)$ and this is a limit, then $f \equiv_{\mathbf{m}} g$.
- If $\nu > 1$ is a successor, there are exactly 4 **m**-equivalence classes in $\{f : \alpha(f) = \nu\}$, arranged as below.

The initial segment of the **m**-degrees includes some recognizable classes.

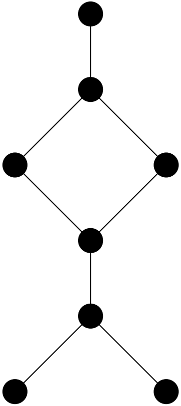


Above Baire 1 – structure of m-degrees

Kihara has shown that the degree structure we found for $\mathcal{B}_1$ continues into higher Baire classes, though α rank was not defined there.

Equivalent definition (Kihara). We have $f \leq_{\mathbf{m}} g$ if and only if for every (p, ε), there is an (r, δ) such that $S_{f,p,\varepsilon} \leq_W S_{g,r,\varepsilon}$, where $\leq_W$ is $\{0, 1, \perp\}$-valued Wadge reducibility and $S_{f,p,\varepsilon}$ is the $\{0, 1, \perp\}$-valued function which outputs the unique acceptable bit for $f(X), p, \varepsilon$, if it exists, or $\perp$ if both are ok.

Using this, he described precisely the structure of the $\leq_{\mathbf{m}}$ degrees, above Baire 1, and their relation to the Wadge degrees.



Thank you.