

FINDING BASES OF UNCOUNTABLE FREE ABELIAN GROUPS IS USUALLY DIFFICULT

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ABSTRACT. We investigate effective properties of uncountable free abelian groups. We show that identifying free abelian groups and constructing bases for such groups is often computationally hard, depending on the cardinality. For example, we show, under the assumption $V = L$, that there is a first-order definable free abelian group with no first-order definable basis.

1. INTRODUCTION

How complicated is it to find a basis of a free abelian group? Can it be done recursively, as we do when building bases for vector spaces? Here by a basis we mean a subset which is both linearly independent and spans the whole group (with integer, rather than rational coefficients). The difficulty is that unlike vector spaces, free abelian groups can contain maximal linearly independent subsets which are not bases. For countable groups, Downey and Melnikov [5] defined a strengthening of linear independence which allows us to recover a recursive construction. The following notion generalises p -independence, which is widely used in the study of torsion-free abelian groups. Recall that a subgroup H of a torsion-free abelian group G is *pure* if $G \cap \mathbb{Q}H = H$; that is, if for all $n \in \mathbb{Z}$ and all $h \in H$, if n divides h in G then it also divides it in H .

Definition 1.1 ([5]). Let G be a torsion-free abelian group. A subset $A \subseteq G$ is *P -independent* if it is linearly independent and its span is a pure subgroup of G .

Using classical techniques, Downey and Melnikov proved that finite P -independent subsets can be recursively extended to bases. Note that any subset of a P -independent set is also P -independent. Let $\mathbb{Z}^\omega = \bigoplus_{k \in \mathbb{N}} \mathbb{Z}$ denote the countably generated free abelian group.

Proposition 1.2 ([5]). *Suppose that $B \subset \mathbb{Z}^\omega$ is a finite P -independent subset; let $g \in \mathbb{Z}^\omega$. Then there is a finite P -independent $B' \supseteq B$ such that $g \in \text{Span}(B)$.*

Again, to be specific, $\text{Span}(B) = \mathbb{Z}B$ is the set of elements of G of the form $\sum m_i a_i$ where $a_i \in B$ and $m_i \in \mathbb{Z}$; B is a basis of \mathbb{Z}^ω if it is linearly independent and spans \mathbb{Z}^ω , if and only if $\mathbb{Z}^\omega = \bigoplus_{b \in B} \mathbb{Z}b$. Of course every basis of \mathbb{Z}^ω must be P -independent. Proposition 1.2 tells us that a basis for \mathbb{Z}^ω can be built recursively, repeatedly extending finite P -independent subsets while ensuring that the next element of the group (in some arbitrary ω -enumeration of the elements of the group) belongs to the span of the basis that we are building.

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Can this process be mimicked when we are given an uncountable free abelian group? We know that there is no important difference between countable and uncountable vector spaces. A basis for a vector space can be built by transfinite recursion, extending as usual at successor steps and taking unions at limit stages. Searching the literature, we found no such construction for uncountable free abelian groups. The purpose of this paper is to show that in most cases, such a construction cannot be performed. One key point is that Proposition 1.2 heavily relies on the fact that B is finite.¹ A recursive construction can get stuck at a limit stage: we can find elements a_1, a_2, \dots , of a free abelian group G such that each finite initial segment $\{a_1, a_2, \dots, a_n\}$ can be extended to a basis of G , but such that the countable set $\{a_1, a_2, \dots\}$ cannot be extended to a basis of G .

Example 1.3 (Melnikov). Let $G = \mathbb{Z}^{\omega+1}$ be a copy of the countably generated free abelian group, with a basis reordered in order-type $\omega + 1$: let $\{e_0, e_1, e_2, \dots, e_\omega\}$ be a basis of G . For $i < \omega$ let $u_i = p_i e_i + e_\omega$, where p_0, p_1, \dots is an enumeration of the prime numbers.

The set $U = \{u_0, u_1, \dots\}$ is P -independent: if $p_j \mid \sum a_i p_i e_i + \sum a_i e_\omega$ then $p_j \mid a_i p_i$ and $p_j \mid \sum a_i$; it follows that $p_j \mid a_i$ for all $i \neq j$, and so must also divide a_j . Proposition 1.2 implies that any finite subset of U can be extended to a basis of G . However, U cannot be extended to a basis of G . Suppose otherwise. Extending to a basis and taking a finite subset, we can find a finite set $V \subset G$ such that $e_\omega \in \text{Span}(V)$ and $V \cup U$ is P -independent. There is some n such that V is spanned by $\{e_0, e_1, \dots, e_\omega\} \setminus \{e_n\}$. Let $H = \text{Span}(V \cup \{u_n\})$. We show that H is not pure, contradicting the P -independence of $V \cup U$. This is because $p_n e_n = u_n - e_\omega$ is an element of H ; but $e_n \notin H$. In fact for any $h \in H$, if $h = \sum_{i \leq \omega} \alpha_i e_i$ then $p_n \mid \alpha_n$; u_n is the only generator that can contribute anything in the n^{th} standard coordinate.²

Of course, one could imagine that there is another property, even stricter than P -independence, adherence to which will allow us to pass limit stages without breaking down. We show that there cannot be any such property.

What do we actually mean by that statement? If G is a free abelian group then there is a property Q of subsets of G (say of smaller cardinality than G) such that:

- Every subset satisfying Q is linearly independent (or even P -independent);
- The analogue of Proposition 1.2 holds: for every subset A satisfying Q and every $g \in G$ there is some $A' \supseteq A$ satisfying Q such that $g \in \text{Span}(A')$; and
- If $A = \bigcup_{\alpha < \lambda} A_\alpha$ is a union of subsets satisfying Q , then A satisfies Q as well.

Namely, we can let Q hold of the subsets of a fixed basis B of G . And in turn, we can use Q to “recursively” build B . What we mean by the statement above is that there is no way to obtain such a property Q if we are just handed the group table for G and don’t have a basis to begin with. Informally, we want to show that it is impossible to only use the group operation of a free abelian group to build a basis.

¹This is a common theme in the investigation of effective properties of uncountable objects: there is a significant difference between finiteness and boundedness. See for example [12, 14].

²More formally: $h = bu_n + \sum b_i v_i$ for $v_i \in V$, and for each i , $v_i = \sum_{j \leq \omega} c_{i,j} e_j$, with $c_{i,n} = 0$ for all i .

To make this statement formal we use the tools of mathematical logic, in particular definability and computability. We fix an uncountable cardinal κ and show that

- For any Δ_1^1 κ -Turing degree \mathbf{d} , there are κ -computable free abelian groups G with no \mathbf{d} -computable basis. In particular, there is a first-order definable free abelian group of size κ with no first-order definable basis.

Before we explain further, we state two important caveats. The first is that this non-definability result holds for most cardinals κ but not for all of them. It is known to fail at some singular cardinals, such as \aleph_ω . Even among regular cardinals, we cannot show this for weakly compact ones. The second caveat is that throughout we make a non-trivial set-theoretic assumption: that all sets are constructible. While this is often harmless when uncountable computability is concerned, it does leave open the possibility that the picture is different under other, possibly strong, set-theoretic assumptions.

Groups in computable algebra and set theory. The study of effective procedures in group theory goes back to work of Max Dehn [4] on finitely presented groups, and in fields, rings and vector spaces to work of Hermann [18], van der Waerden [30], and explicitly using computability to Rabin [26], Maltsev [23], Fröhlich and Shepherdson [10], and Metakides and Nerode (for example [24]). The basic idea is to study how effective algebraic objects and processes are. For example, famously, Novikov and Boone (see for example [25, 2]) showed that the word problem in groups may fail to be solved effectively; the same holds for conjugacy and isomorphism questions. Similarly, Higman's embedding theorem [19] characterises embeddability into finitely presented groups using an effective criterion.

The key notion is that of a *computable group*: this is one whose collection of elements is a computable set (say of natural numbers), and the group operation can be performed effectively (computably). Key questions are: (a) which groups have computable copies? and (b) how similar or different are various computable copies of the same group? One possible answer for the second question is encapsulated in the notion of *computable categoricity*, meaning that all computable copies are isomorphic via computable isomorphisms; informally, this means that all computable copies have the same computable properties. For example, finitely generated free abelian groups are computably categorical, since a bijection between two finite bases effectively lifts to an isomorphism of the groups.

Very few groups are computably categorical, and so it makes sense to consider weakenings of this notion by allowing the help of the jump operator. For example, we say that a group is Δ_2^0 -categorical if any two computable copies are isomorphic via a Δ_2^0 ($\mathbf{0}'$ -computable) isomorphism. For a free abelian group, the complexity of isomorphisms with a standard computable copy (with a computable basis) is the same as the complexity of bases. In [5], Downey and Melnikov use Proposition 1.2 to show that the countably generated free abelian group is Δ_2^0 -categorical, equivalently, that every computable copy of the countably generated free abelian group has a Δ_2^0 basis.³

Uncountable free groups were studied by algebraists and set theorists. Best known is Shelah's work on the Whitehead problem [28]. Two main questions which

³This is sharp: in this paper we show that there is a computable copy of \mathbb{Z}^ω , every basis of which computes $\mathbf{0}'$.

were addressed are: (1) for which cardinals λ are there λ^+ -free groups which are not λ -free? (2) Is it possible to axiomatise the class of free abelian groups in infinitary logic? The latter question is related to results below on the complexity of the collection of free abelian groups of a fixed cardinality. Some techniques used for the investigation of these questions are related to ones we use below. These investigations though were not concerned with questions of effectiveness. See for example [20, 6, 29].

Uncountable computable algebra. The tools of traditional computability are restricted to investigating countable groups, since the basic objects that can be manipulated by computers are hereditarily finite. To be able to make sense of the questions above for uncountable groups we use an extension of computability to uncountable domains. Several approaches have been suggested (see [13]). In this paper we use *admissible computability*, as described in [16], to investigate uncountable computable model theory. This approach was successfully used in [14, 15] to investigate uncountable linear orderings. An abstract investigation of computable categoricity in this setting is given in [3, 17].

There are several ways to describe admissible computability. Köpke [22] used Turing machines with transfinite tapes. The original way, and quickest, is to use definability. Let κ be a cardinal. The universe for κ -computability is $H(\kappa)$, the collection of all sets whose transitive closure is of size smaller than κ . A set is defined to be κ -c.e. if it is Σ_1^0 -definable over $H(\kappa)$ (with parameters). A set is κ -computable if it is both κ -c.e. and co- κ -c.e.; a function is partial κ -computable if its graph is κ -c.e. The main assumption which makes computability work is that there is a κ -computable isomorphism between κ and the universe $H(\kappa)$. Most commonly this is achieved by assuming that every set in $H(\kappa)$ is constructible, in which case $H(\kappa) = L_\kappa$. Note that this holds for $\kappa = \omega$, and that ω -computability is the familiar notion of Turing computability. The main tool in κ -computability is defining computable functions recursively. Formally, if $I: H(\kappa) \rightarrow H(\kappa)$ is κ -computable then there is a unique function $f: \kappa \rightarrow H(\kappa)$ such that for all $\alpha < \kappa$, $f(\alpha) = I(f \upharpoonright_\alpha)$; this function is κ -computable. The main point is that even when κ is singular, $f \upharpoonright_\alpha \in H(\kappa)$; we say that $H(\kappa)$ is *admissible*.

For more details on κ -computability see [27, 16]. As we mentioned above, throughout this paper we assume that $V = L$.

Identifying free groups. When investigating the complexity of free abelian groups we come across a closely related question: how complicated is it to tell if a given (torsion-free and abelian) group is free? Informally, the idea is that if there were some effective or definable way to take the group operation of a free group and produce a basis, we could start with any group, attempt to build a basis according to this procedure, and see if we succeed or fail. For example, for countable groups this approach, using the Downey-Melnikov procedure described above, gives an upper bound for the complexity of the collection of free groups (it is Π_3^0 , in fact complete at that level). On the other hand it stands to reason that a procedure that tells whether a given group is free can be used to get a proof of this fact, namely a basis. Thus the complexity of the two problems, of identifying free groups, and of building bases, is often related. We shall see though that in some cases this intuition does not seem to hold.

There is a natural upper bound to the complexity of the collection of free groups, namely Σ_1^1 — the defining formula is “the group has a basis”. A proof that this collection is Σ_1^1 -complete would show that there is no simpler way of identifying free groups; a proof that this collection is much simpler (say first-order definable) would show that there is some kind of effective or definable procedure to find out whether a group is free, without having to divine a basis out of nowhere. Our first theorem settles the complexity of the collection of free abelian groups for regular uncountable cardinals.

Theorem 1.4 ($V = L$). *Let κ be a regular uncountable cardinal.*

- (1) *If κ is not weakly compact then the collection of free abelian groups is Σ_1^1 -complete. If further κ is a successor cardinal, or the least inaccessible cardinal, then this collection is Σ_1^1 -complete.*
- (2) *If κ is weakly compact then the collection of free abelian groups is Π_1^0 -complete (indeed it is $\Pi_1^0(\emptyset')$ -complete) in the set of groups. The index-set of the computable free abelian groups is Π_2^0 -complete.*

We should be more formal about what we actually mean. Just as for $\kappa = \omega$, if κ is regular then we can discuss the complexity of subsets of 2^κ (or κ^κ) using definability. The subset of 2^κ defined by a formula φ (in the language of set theory) is the collection of $A \subseteq \kappa$ such that the structure $(H(\kappa); \in, A)$ satisfies φ . We allow quantification over subsets of κ ; for example, a Σ_1^1 formula $\exists X \varphi$ holds of A if there is some $B \subseteq \kappa$ such that $(H(\kappa); \in, A, B) \models \varphi$. In all of these formulas we allow parameters from the structure $H(\kappa)$. We also use the usual conventions for boldface classes, to denote that we allow a fixed subset predicate.

Lightface statements of completeness are effective. We use a rich topology for 2^κ : basic open sets are specified by specifying fewer than κ bits. A partial continuous function from 2^κ to itself is defined by a *functional*, a set Φ of pairs (p, q) where $p, q \in 2^{<\kappa}$; the defined function maps $A \in 2^\kappa$ to $\Phi(A)$ defined by $q < \Phi(A)$ if and only if there is some $p < A$ such that $(p, q) \in \Phi$. If Φ itself is κ -c.e. then the induced function is called partial κ -computable. We remark that just as the case $\kappa = \omega$, relative κ -computability can be defined using these maps; we say that $X \in 2^\kappa$ is κ -computable from $Y \in 2^\kappa$ if $f(Y) = X$ for some partial κ -computable f .

What Σ_1^1 completeness means is that for any Σ_1^1 set R there is a κ -computable function $f: 2^\kappa \rightarrow 2^\kappa$ such that for all $A \in 2^\kappa$, $A \in R$ if and only if $f(A)$ is (the graph of the group operation of) a free abelian group. This also gives an index-set result: it shows that the collection of indices of partial κ -computable functions which are total and compute (the graph of the group operation of) a free abelian group is complete among all Σ_1^1 subsets of κ .

By boldface completeness we mean to allow an oracle. That is, Σ_1^1 -completeness stated above says that there is some $A \subseteq \kappa$ such that the collection of free abelian groups is $\Sigma_1^1(A)$, and for any $\Sigma_1^1(A)$ set R there is an A -computable function f which reduces R to the collection of free abelian groups.

We also remark that the first part of Theorem 1.4 can be relativised to any oracle. Namely, if the collection of free abelian groups of size κ is $\Sigma_1^1(A)$ -complete for some $A \in 2^\kappa$, then it is also $\Sigma_1^1(B)$ -complete for all $B \in 2^\kappa$ which κ -compute A .

We remark though that when we later discuss singular cardinals we cannot relativise to any oracle, as for many oracles A the structure $(H(\kappa); \in, A)$ will not be admissible.

The complexity of bases. Theorem 1.4 gives us information about the complexity of bases of free groups. The fact that there is a complete Σ_1^1 subset of κ implies:

Corollary 1.5 ($V = L$). *Let κ be a successor cardinal. For any Δ_1^1 set $X \in 2^\kappa$ there is a κ -computable free abelian group which has no X -computable basis.*

We note that the class $\Delta_1^1(L_\kappa)$ is huge. It properly contains all κ -hyperarithmetical sets (under any reasonable definition of that concept), all sets in the least admissible set beyond L_κ (or the least model of ZF^-), and more.

One could hope for more. Can we not only avoid lower cones but code complicated information into all bases of a group? We will show that this is not the case; bases can be built by forcing and so can avoid computing even simple sets.

Theorem 1.6 ($V = L$). *Let κ be a regular uncountable cardinal singular cardinal.*

- *If κ is a successor of a regular uncountable cardinal which is not weakly compact, let $D = \emptyset''$ (the complete $\Sigma_2^0(L_\kappa)$ set).*
- *Otherwise let $D = \emptyset'$ (the complete $\Sigma_1^0(L_\kappa)$ set).*

Then:

- (1) *There is a κ -computable free abelian group, all of whose bases κ -compute D .*
- (2) *If $X \not\leq_\kappa D$ then every κ -computable free abelian group has a basis which does not κ -compute X .*

In fact coding in \emptyset' is not complicated; we will show that for any cardinal κ there is a κ -computable free abelian group, all of whose bases compute \emptyset' . The proof covers $\kappa = \omega$ and singular cardinals as well.

Singular cardinals. Singular cardinals pose many difficulties. If κ is singular then for many sets A , (L_κ, A) is not admissible, and computability itself behaves in strange ways. For example, the \aleph_{ω_1} -degrees above \emptyset' are well-ordered [9]. However L_κ itself is admissible and in some cases we can say something about κ -computable groups. For example, we can code \emptyset' into bases of a group. In the case of cofinality ω , the complexity introduced by closed and unbounded sets disappears, and we can say more.

Theorem 1.7 ($V = L$).

- (1) *Every \aleph_ω -computable group has a \emptyset' -computable basis.*
- (2) *The index set of the \aleph_ω -computable free abelian groups is Π_2^0 -complete.*

A more general theorem holds for all cardinals of cofinality ω .

Questions. We are left with several questions.

- (1) Can Corollary 1.5 be strengthened? For example, is there a κ -computable group with no $\Delta_1^1(L_\kappa)$ basis? We remark that for regular uncountable cardinals there is no “overspill” phenomenon.
- (2) There are cases which were not covered. For example, we don’t know if Corollary 1.5 holds for weakly compact cardinals.
- (3) What happens if $V \neq L$? Recall that for computability to take a familiar form we assume that there is a κ -computable bijection between κ and the universe $H(\kappa)$. For $\kappa = \aleph_1$ this implies that all reals are constructible, but it is consistent with some subsets of ω_1 not being constructible. For

$\kappa = \aleph_2$ this is a consequence of some forcing axioms (for example PFA), which imply the failure of CH.

- (4) What can be said about more complicated groups? Some of the results can be extended to homogeneously completely decomposable groups (see [5]). It may be interesting to investigate the effective properties of uncountable such groups.

2. IDENTIFYING FREE GROUPS

In this section we will prove Theorem 1.4. We will first consider successor cardinals and later deal with the inaccessible cases. We start with a few basic facts. Recall that throughout this paper, we assume that $V = L$. A general reference for torsion-free abelian groups is [11].

2.1. Detachment, freeness, and clubs. All groups we will discuss are abelian and torsion-free. A group G is *free abelian* if it has a basis: a subset B which is linearly independent ($\sum m_i b_i = 0$ implies each $m_i = 0$, where $m_i \in \mathbb{Z}$ and $b_i \in B$) and spans G (every element of G is of the form $\sum m_i b_i$ for some $m_i \in \mathbb{Z}$ and $b_i \in B$). We will omit the adjective “abelian” and just call these groups free. For any infinite cardinal κ , the free group of size κ will be denoted by \mathbb{Z}^κ .

Fact 2.1. Any subgroup of a free group is free.

The following is a key notion.

Definition 2.2. If G is a group and $H \subseteq G$ is a subgroup, we say that H *detaches in* G if $G = H \oplus K$ for some subgroup $K \subseteq G$. We write $H \mid G$.

If G is free then $H \mid G$ if and only if some basis of H can be extended to a basis of G if and only if every basis of H can be extended to a basis of G .

Fact 2.3. Suppose that G is free and that $H \subseteq G$ is a subgroup. Then H detaches in G if and only if the quotient group G/H is free.

If H is a subgroup of a group G then we write $[H, G]$ to denote the interval in the lattice of subgroups: it is the collection of all subgroups $K \subseteq G$ such that $H \subseteq K$.

Proposition 2.4. *Suppose that H detaches in G . Then H detaches in every subgroup $K \in [H, G]$.*

Proof. Suppose that $G = H \oplus G'$ for some $G' \subseteq G$. Let $K' = K \cap G'$. Then $K = H \oplus K'$. For if $g \in K$, then $g \in G$, so $g = h + k$ where $h \in H$ and $k \in G'$ and this decomposition is unique. Since $h, g \in K$, we have $k \in K$, so $k \in K'$. \square

A sequence $\langle G_\alpha \rangle_{\alpha < \gamma}$ of groups of some ordinal length γ is *increasing* if $\alpha < \beta$ implies $G_\alpha \subseteq G_\beta$; it is *continuous* if for all limit $\alpha < \gamma$, $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$. A *filtration* of a group G is a sequence $\bar{G} = \langle G_\alpha \rangle$ such that $\bar{G} \langle G \rangle$ is increasing and continuous, and $|G_\alpha| \leq |\alpha|$ for all α .

If γ is regular and G is a group of universe γ then all filtrations of G agree on a club; in fact, for club many α , $G_\alpha = G \cap \alpha$. We decide that the *standard filtration* of a group G of universe a regular cardinal γ is defined by $G_\alpha = \text{Span}(G \cap \alpha)$.

Definition 2.5. Let $\bar{G} = \langle G_\alpha \rangle_{\alpha < \gamma}$ be increasing and continuous. The *detachment set* of \bar{G} is

$$\text{Div}(\bar{G}) = \{\alpha < \gamma : \forall \beta \in (\alpha, \gamma) (G_\alpha \mid G_\beta)\}.$$

If γ is regular and \bar{G}, \bar{G}' are two filtrations of a group of universe γ , then $\text{Div}(\bar{G})$ and $\text{Div}(\bar{G}')$ agree on a club; this uses Proposition 2.4. In this case we will write $\text{Div}(G)$ for $\text{Div}(\bar{G})$, where \bar{G} is the standard filtration of G .

Proposition 2.6. Let γ be a limit ordinal and let $\bar{G} = \langle G_\alpha \rangle_{\alpha < \gamma}$ be a filtration of a group G_γ . Suppose that for all $\alpha < \gamma$, G_α is free.

- (1) If $\text{Div}(\bar{G})$ contains a club of γ then G_γ is free.
- (2) If γ is a regular cardinal and G_γ is free then $\text{Div}(\bar{G})$ contains a club of γ .

Proof. For (2), let B be a basis for G_γ . There are club many $\alpha < \gamma$ for which $G_\alpha = \text{Span}(B \cap \alpha)$; each such α belongs to $\text{Div}(\bar{G})$.

For (1), suppose that $C \subseteq \text{Div}(\bar{G})$ is closed and unbounded. We may assume that $G_0 = 0$ and that $0 \in C$. For $\alpha \in C$ let $\alpha' = \min C \setminus (\alpha + 1)$ be the next element of C after α . If $\alpha \in C$ then $G_\alpha \mid G_{\alpha'}$; choose some H_α such that $G_{\alpha'} = G_\alpha \oplus H_\alpha$. Then $G_\gamma = \bigoplus_{\alpha \in C} H_\alpha$. Each H_α is free (as $G_{\bar{\alpha}}$ is free). If B_α is a basis of H_α , then $\bigcup_{\alpha \in C} B_\alpha$ is a basis of G_γ . \square

If γ is a regular cardinal and $\langle G_\alpha \rangle$ is a filtration of a group G of universe γ , then the relation $G_\alpha \mid G_\beta$ is γ -c.e.; we need to search for a complement for G_α in G_β (when G_β is free, equivalently we search for a basis of G_β/G_α). We will see that for some γ this relation will be $\Sigma_1^0(L_\gamma)$ -complete, but for other γ the relation will be γ -computable. Note that the standard filtration of G is G -computable.

Remark 2.7. Let γ be a limit ordinal; let $\bar{G} = \langle G_\alpha \rangle_{\alpha < \gamma}$ be a filtration of a group G_γ . Suppose that $\text{Div}(\bar{G})$ contains a club of γ . Then

$$\text{Div}(\bar{G}) = \{\alpha < \gamma : G_\alpha \mid G_\gamma\}.$$

One direction follows from Proposition 2.4; the other from the proof of Proposition 2.6.

2.2. Σ_1^1 completeness of finding clubs, and a class arising from the proof of square principles. We saw that identifying a free group reduces to finding club subsets of the definable set $\text{Div}(G)$. Thus, our stated result would imply that existence of a club subset is Σ_1^1 -complete. This is indeed the case; this was proved for $\kappa = \omega_1$ by Fokina et al. in [8]. The proof generalises. We will need this fact and will need to get more information from its proof.

Here is a key notion.

Definition 2.8. For a singular ordinal α , we let $s(\alpha)$ be the least ordinal $\beta \geq \alpha$ such that there is a cofinal sequence in α of order-type smaller than α which is definable over J_β .

In other words this is the first place at which we recognise that α is singular. The sets J_β are Jensen's modification of the L_α hierarchy which is required to make fine structure theory work (the sets J_β are closed under the rudimentary functions). The details are unimportant, and for sufficiently nice ordinals α we have $L_\alpha = J_\alpha$ anyway. We will use some basic facts which hold for both hierarchies. For example, the function $\alpha \mapsto J_\alpha$ is Σ_1 -definable in every J_β for $\beta > \alpha$. Also, the subsets

of J_β which are elements of $J_{\beta+1}$ are precisely the ones definable over J_β (with parameters).

We note:

- The function $\alpha \mapsto s(\alpha)$ is Σ_1 -definable, and so its restriction to ordinals below a cardinal κ is partial κ -computable.

The domain of this function, the set of singular ordinals below a cardinal κ , may fail to be κ -computable; it is merely κ -c.e. Note that this only happens when κ is a limit cardinal. If κ is a successor cardinal then the restriction of the function $\alpha \mapsto s(\alpha)$ to ordinals below κ is κ -computable.

The following class was defined by Jensen in [21, Sec.5], the original paper on fine structure.

Definition 2.9. The class E consists of all the singular ordinals α such that for some $\beta \in (\alpha, s(\alpha))$:

- $J_\beta \models \text{ZF}^-$;
- α is the greatest cardinal of J_β ;
- for some $p \in J_\beta$, J_β is the least (fully) elementary substructure $M < J_\beta$ such that $p \in M$ and $M \cap \alpha$ is transitive.

Suppose that $\alpha \in E$ and let $\beta > \alpha$ witness this fact. Then J_β can be presented as the countable union $\bigcup M_i$, with $M_0 = \{p\}$ and each M_{i+1} being the $\Sigma_i(J_\beta)$ -Skolem hull of $M_i \cup \text{sup}(M_i \cap \alpha)$. The sequence $\langle M_i \rangle$ is definable over $J_{\beta+1}$. However, for all i , the process generating M_i is definable over J_β . Since $\beta < s(\alpha)$, $M_i \cap \alpha$ is bounded below α . This implies that:

- Each $\alpha \in E$ has countable cofinality, and $s(\alpha) = \beta + 1$.

The definition of E was designed to ensure the following:

Lemma 2.10. *Let κ be regular and uncountable; let $q \in L_{\kappa^+}$. Let M be the least elementary substructure of L_{κ^+} such that $q \in M$ and $M \cap \kappa$ is transitive. Let $\pi: M \rightarrow J_\beta$ be the Mostowski collapse; let $\alpha = \pi(\kappa) = M \cap \kappa$. Then $\alpha \in E$, witnessed by β .*

The main idea, for showing that $\beta < s(\alpha)$, is that if $\gamma < \alpha$ and $f: \gamma \rightarrow \alpha$ is J_β -definable and cofinal, then the same definition over M (equivalently L_{κ^+}) gives a cofinal $\hat{f}: \gamma \rightarrow \kappa$, which is impossible. It follows that if κ is regular, then $E \cap \kappa$ is stationary in κ : for any club C of κ , consider the least elementary $M < L_{\kappa^+}$ such that $C \in M$ and $M \cap \kappa$ is transitive. A similar argument gives the Σ_1^1 -completeness of containing a club. We will make use of the following tool.

Definition 2.11. Let κ be regular and uncountable, let $B \in 2^\kappa$, and let $\forall X \varphi$ be a Π_1^1 formula, where φ is first-order with parameter $r \in L_\kappa$. We let $F(B, \varphi)$ be the set of singular ordinals $\alpha < \kappa$ such that $r \in J_\alpha$, $B \upharpoonright_\alpha \in J_{s(\alpha)}$ and for all $X \in J_{s(\alpha)}$, $(J_\alpha, B \upharpoonright_\alpha, X) \models \varphi$.

That is, $\alpha \in F(B, \varphi)$ if we believe that the Π_1^1 property under discussion holds of $(J_\alpha, B \upharpoonright_\alpha)$, where we limit the second-order quantifiers to subsets of α which are only constructed at stages at which we still think that α is regular.

Lemma 2.12. *Let κ be a regular cardinal, let $B \in 2^\kappa$, and let $\forall X \varphi$ be a Π_1^1 formula.*

- (1) *If $(L_\kappa, B) \models \neg \forall X \varphi$ then $F(B, \varphi)$ is nonstationary in κ .*
- (2) *If $(L_\kappa, B) \models \forall X \varphi$ then $E \cap F(B, \varphi)$ is stationary in κ .*

Proof. Let r be the parameter for φ .

For (1), we build a continuous and increasing sequence $\langle M_i \rangle_{i < \kappa}$ of elementary submodels of L_{κ^+} such that $r, B \in M_0$ and $\alpha_i = \kappa \cap M_i$ is an element of κ ; the set $\{\alpha_i : i < \kappa\}$ is closed and unbounded in κ (we let $\alpha_i \in M_{i+1}$). Let $\pi_i : M_i \rightarrow J_{\beta_i}$ be the Mostowski collapse. The argument above shows that $\beta_i < s(\alpha_i)$. There is some $X \in M_0 \cap 2^\kappa$ such that $(L_\kappa, B, X) \models \neg\varphi$; then $X \upharpoonright_{\alpha_i} = \pi_i(X) \in J_{\beta_i}$ (and $B \upharpoonright_{\alpha_i} \in J_{\beta_i}$) and $(J_{\alpha_i}, B \upharpoonright_{\alpha_i}, X \upharpoonright_{\alpha_i}) \models \varphi$ (as J_{β_i} thinks it does, and this is absolute). Hence the club $\{\alpha_i : i < \kappa\}$ is disjoint from $F(B, \varphi)$.

For (2), let C be a club of κ . Let $M < L_{\kappa^+}$ be least such that $r, C, B \in M$ and $M \cap \kappa \in \kappa$. Let $\pi : M \rightarrow J_\beta$ be the Mostowski collapse and let $\alpha = \pi(\kappa) = M \cap \kappa$. Then $\alpha \in E \cap C$, and $s(\alpha) = \beta + 1$. Suppose that $X \in J_{\beta+1} \cap 2^\alpha$. It is definable over J_β , say with parameter q . Let \hat{X} be the interpretation of the same definition over M (equivalently L_{κ^+}), with parameter $\pi^{-1}(q)$. Then $(L_\kappa, B, \hat{X}) \models \varphi$. It follows that $(J_\alpha, B \upharpoonright_\alpha, X) \models \varphi$, so $\alpha \in F(B, \varphi)$.⁴ \square

Corollary 2.13. *Let κ be a successor cardinal. The nonstationary ideal on κ (equivalently the club filter on κ) is Σ_1^1 -complete. In fact, the restriction of the nonstationary ideal to $E \cap \kappa$ is Σ_1^1 -complete. That is, for any $\Sigma_1^1(L_\kappa)$ set $A \subseteq 2^\kappa$ there is a κ -computable function $f : 2^\kappa \rightarrow 2^\kappa$ such that for all $Y \in 2^\kappa$, $f(Y) \subseteq E$, and $Y \in A$ if and only if $f(Y)$ is nonstationary.*

Proof. Let $\exists X \varphi$ be the formula defining A ; we let $f(Y) = E \cap F(Y, \neg\varphi)$. Recall that the set of singular ordinals below κ is κ -computable; this implies that $E \cap \kappa$ is κ -computable and that $F(Y, \neg\varphi)$ is Y -computable, uniformly in Y . \square

A key fact that we will use for $\kappa \geq \aleph_2$ is the following, which is [21, Thm.5.1].

Theorem 2.14 (Jensen). *The class E does not reflect at any singular ordinal. That is, if α is singular then $E \cap \alpha$ is nonstationary in α .*

The proof of this fact is complicated. It is part of the proof of the square principle in L .

2.3. Twisting a group. The plan for proving Theorem 1.4 for the case of successor cardinals is to take a set $Y \subseteq \kappa$ and produce a Y -computable group G such that $\text{Div}(G) = \kappa \setminus f(Y)$, where f is given by Corollary 2.13. A main tool would be to take a group G_α which we have already constructed, and ensure that it does not detach in G by ensuring that it does not detach in $G_{\alpha+1}$. On the other hand we need to ensure that for all $\beta < \alpha$, if we already declared that we want G_β to detach in G , then G_β detaches in $G_{\alpha+1}$. We need to “twist” G_α without further twisting any G_β for $\beta < \alpha$.

The idea is to use Melnikov’s Example 1.3. Using the notation of that example, we observe that the set U is P -independent, as it is the union of P -independent sets. Hence U is a basis of the span of U inside $\mathbb{Z}^{\omega+1}$, a group isomorphic to \mathbb{Z}^ω . However the span of U does not detach inside $\mathbb{Z}^{\omega+1}$; adding a basis for a complement would give an extension of U to a basis of $\mathbb{Z}^{\omega+1}$, which we showed does not exist. On the other hand, for all n , $H_n = \text{Span}(u_0, \dots, u_n)$ does detach in $\mathbb{Z}^{\omega+1}$, because $\{u_0, \dots, u_n\}$ is finite and P -independent, and so extends to a basis of $\mathbb{Z}^{\omega+1}$ (Proposition 1.2). Turning this around, if we start with some copy

⁴In the definition of $F(B, \varphi)$ we could replace $s(\alpha)$ by $s(\alpha) - 1$, assuming that we are restricting ourselves to $\alpha \in E$.

of \mathbb{Z}^ω , say with some basis $\{v_0, v_1, \dots\}$, then we can embed it into $\mathbb{Z}^{\omega+1}$ by mapping each v_i to u_i ; the original \mathbb{Z}^ω is now “twisted” inside $\mathbb{Z}^{\omega+1}$, but finitely generated subgroups are not.

We generalise this to possibly uncountable groups by picking out countable pieces.

Proposition 2.15. *Suppose that $\langle H_n \rangle$ is an increasing sequence of free groups such that for all n , $H_n \mid H_{n+1}$; so $H_\omega = \bigcup_n H_n$ is free as well. There is a free group G extending H_ω (with $|G| = |H_\omega|$) such that $H_\omega \nmid G$ but for all n , $H_n \mid G$. The group G can be obtained effectively from the sequence $\langle H_n \rangle$.*

We write $\text{twist}(\langle H_n \rangle)$ for the group G .

Proof. Without loss of generality we assume that H_0 is the trivial group. As in the proof of Proposition 2.6, choose subgroups K_n such that $H_{n+1} = H_n \oplus K_n$, so $H_\omega = \bigoplus_n K_n$. As each K_n is free, we write $K_n = P_n \oplus Q_n$, where $P_n \cong \mathbb{Z}$, say generated by some b_n . Let $P = \bigoplus_n P_n$ and $Q = \bigoplus_n Q_n$.

Using Melnikov’s example, as discussed above, and renaming, we can find a free group $R \supseteq Q$, $R \cong \mathbb{Z}^{\omega+1}$, such that $Q \nmid R$, but for any n , $Q_{<n} = \bigoplus_{m<n} Q_m$ does detach inside R . We let $G = P \oplus R$.

It follows that for all n , $P \oplus Q_{<n}$ detaches in G . As H_n detaches in $P \oplus Q_{<n}$, and detachment is transitive, we see that each H_n detaches in G .

It also follows that $H_\omega = P \oplus Q$ does not detach in G ; if $H_\omega \mid G$ then $Q \mid G$ and as $Q \subseteq R \subseteq G$ we would have $Q \mid R$ (Proposition 2.4). \square

2.4. The successor case. We prove Theorem 1.4 where κ is a successor cardinal:

Theorem 2.16. *Let κ be a successor cardinal. The set of free abelian groups of universe κ is $\Sigma_1^1(L_\kappa)$ -complete.*

Proof. Let $A \subseteq 2^\kappa$ be Σ_1^1 . Given $Y \in 2^\kappa$ we (uniformly) compute an abelian group G_Y which is free if and only if $Y \in A$. To begin, we find a set $U = U_Y \subseteq E \cap \kappa$, effectively obtained from Y , such that $Y \in A$ if and only if U is nonstationary (Corollary 2.13). Without loss of generality, $U \cap \kappa^- = \emptyset$, where κ^- is the cardinal predecessor of κ . The point here is that every ordinal in (κ^-, κ) is singular, and so E does not reflect at any ordinal in this interval (Theorem 2.14).

We will build a filtration $\langle G_\alpha \rangle_{\alpha < \kappa}$ of a group G_Y such that each G_α is free, and $\text{Div}(\langle G_\alpha \rangle) = \kappa \setminus U$; our desired equivalence then holds by Proposition 2.6.

We define the sequence $\langle G_\alpha \rangle$ computably in U . The construction is of course by recursion on α . For the construction to make sense, simultaneously with the construction we ensure that:

- (i) each G_α is free; and
- (ii) for all $\beta < \alpha$, if $\beta \notin U$ then $G_\beta \mid G_\alpha$.

We start with G_0 being the trivial group. Taking $\alpha < \kappa$, we assume that G_β is defined for all $\beta < \alpha$ and that the inductive hypothesis holds below α .

Case 1: α is a successor ordinal and $\alpha - 1 \notin U$. We let $G_\alpha = G_{\alpha-1} \oplus \mathbb{Z}$. (i) for α holds easily. (ii) holds because $G_{\alpha-1} \mid G_\alpha$ and detachment is transitive.

Case 2: α is a successor ordinal and $\alpha - 1 \in U$. Since $\alpha - 1 \in E$ we know that $\text{cf}(\alpha - 1) = \omega$. We can choose an increasing and cofinal sequence $\langle \alpha_n \rangle$ in $\alpha - 1$ which is disjoint from U , for example consisting of successor ordinals. By induction, for all n , $G_{\alpha_n} \mid G_{\alpha_{n+1}}$. We can thus apply Proposition 2.15: we let $G_\alpha = \text{twist}(\langle G_{\alpha_n} \rangle)$.

(i) holds by construction. For (ii), let $\beta < \alpha$, $\beta \notin U$. There is some n such that $\beta < \alpha_n$. By induction, $G_\beta \mid G_{\alpha_n}$; by construction, $G_{\alpha_n} \mid G_\alpha$.

Case 3: α is a limit ordinal. We let $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$. To verify (i) and (ii) in this case we use the fact that there is a club C of α which is disjoint from U (Theorem 2.14). It follows that $\text{Div}(\langle G_\beta \rangle_{\beta < \alpha})$ contains a club, and so G_α is free (Proposition 2.6). (ii) follows from Remark 2.7.

Note that in case 2, to perform the twist, we need a basis for G_α . However we know that G_α is free, so we can simply search for a basis until we find it. Identifying that B is a basis of a free group G is κ -computable.

(ii) above implies that $\kappa \setminus U \subseteq \text{Div}(G_Y)$. However if $\alpha \in U$ then we ensured that $G_\alpha \not\mid G_{\alpha+1}$, so $\alpha \notin \text{Div}(G_Y)$. This completes the proof. \square

2.5. Lightface weak compactness. A cardinal κ is weakly compact if and only if it is Π_1^1 -indescribable. Under $V = L$, for $B \in 2^\kappa$, say that κ is $\Pi_1^1(B)$ -describable if there is a $\Pi_1^1(B)$ fact which holds for L_κ but not for L_λ for any $\lambda < \kappa$ (we may restrict ourselves to regular $\lambda < \kappa$, since this is expressible by a Π_1^1 -statement). That is, if for some first-order φ , for all $X \in 2^\kappa$, $(L_\kappa, B, X) \models \varphi$, but for all $\lambda < \kappa$, for some $X \in 2^\lambda$, $(L_\lambda, B \upharpoonright_\lambda, X) \models \neg\varphi$. For example, the least inaccessible cardinal is Π_1^1 -describable. A cardinal κ is weakly compact if and only if it is $\Pi_1^1(B)$ -indescribable for all $B \in 2^\kappa$. The next part of Theorem 1.4 follows from the following theorem.

Theorem 2.17. *Let κ be an inaccessible cardinal, and let $B \in 2^\kappa$. If κ is $\Pi_1^1(B)$ -describable then the collection of free abelian groups of size κ is $\Sigma_1^1(B)$ -complete.*

Note that if κ is $\Pi_1^1(B)$ -describable and C κ -computes B , then κ is also $\Pi_1^1(C)$ -describable. Also note that Theorem 2.17 implies that the collection of free abelian groups on the least inaccessible cardinal is Σ_1^1 -complete.

2.6. An elaboration on square for inaccessible cardinals. Toward proving Theorem 2.17, we need an elaboration on the class E above and on Corollary 2.13. Consider what would go wrong if we tried to replicate the proof of Theorem 2.16 for an inaccessible cardinal κ . One problem is that $E \cap \kappa$ is no longer κ -computable, merely κ -c.e.; we will need to address this problem in the construction below by approximating the final filtration $\langle G_\alpha \rangle$ while still building a computable group. A more serious obstacle is that E does not reflect at all regular cardinals, and so unboundedly below κ . This would mean that we will not be able to ensure that all the groups G_α that we build along the way are free. We need to restrict ourselves to a sparser class which will be stationary in κ but not reflect (will not be stationary in any $\alpha < \kappa$).

If κ is weakly compact then every stationary subset of κ reflects (being stationary is Π_1^1). Hence there is no hope to perform this construction in this case. And indeed, below we use this very fact to give an easy characterisation of free groups of a weakly compact size. Jensen showed that in L , this is the only problematic case.

Recall Definition 2.11 of the set $F(B, \varphi)$.

Proposition 2.18. *Let κ be inaccessible and $\Pi_1^1(B)$ -describable, say by the formula $\forall X \varphi$. Then $E \cap F(B, \varphi)$ is stationary in κ , but does not reflect at any limit ordinal $\alpha < \kappa$.*

Proof. That $E \cap F(B, \varphi)$ is stationary follows from Lemma 2.12(2).

Let $\alpha < \kappa$ be a limit ordinal. If α is singular, then we know that E does not reflect at α . If α is a regular cardinal then by assumption, $(L_\alpha, B \upharpoonright_\alpha) \models \exists X \neg \varphi$; by Lemma 2.12, $F(B, \varphi)$ is nonstationary in α . \square

For brevity let $F = E \cap F(B, \varphi)$. Replacing E by F in the proof of Lemma 2.12 shows that the nonstationary ideal on κ , in fact its restriction to F , is $\Sigma_1^1(F)$ -complete. Copying the construction proving Theorem 2.16 shows that the collection of free abelian groups of size κ is $\Sigma_1^1(F)$ -complete. However, this does not quite give Theorem 2.17, because F may fail to be B -computable; it is merely B -c.e. As mentioned above, we modify the construction to approximate the desired result.

Proof of Theorem 2.17. Let $F = E \cap F(B, \varphi)$, where κ is $\Pi_1^1(B)$ -describable, witnessed by the formula $\forall X \varphi$. Let A be $\Sigma_1^1(B)$, say defined by the formula $\exists X \psi$. Given $Y \in 2^\kappa$ we let $U = U_Y = F \cap F(Y, \neg\psi)$. If $Y \in A$ then $F(Y, \neg\psi)$ is nonstationary in κ (Lemma 2.12), and so U is nonstationary. If $Y \notin A$ then by the same lemma, U is stationary, as it contains $E \cap F((B, Y), \varphi \wedge \neg\psi)$.

Again our aim is to build a group G_Y of universe κ and a filtration $\bar{G} = \langle G_\alpha \rangle_{\alpha < \kappa}$ of G such that $\text{Div}(\bar{G}) = \kappa \setminus U$. The group G_Y needs to be $Y \oplus B$ -computable, uniformly in Y ; but as mentioned above, the filtration \bar{G} will not.

What we do have, effectively from $Y \oplus B$, is an enumeration of U : an increasing and continuous sequence $\langle U_s \rangle_{s < \kappa}$ of sets in L_κ such that $U = \bigcup_{s < \kappa} U_s$. Namely we let U_s be the collection of $\alpha \in U$ such that $s(\alpha) < s$. Again the point is that the set of singular ordinals below κ is κ -c.e.; once we see that α is singular we can effectively, from $B \oplus Y$, check whether $\alpha \in U$ or not. Note that this means that for any cardinal $\lambda < \kappa$, $U_\lambda = U \cap \lambda$. For all s , $U_s \subseteq s$.

We do define G_Y by building a $Y \oplus B$ -computable increasing sequence $\langle H_s \rangle$. The problem with copying the previous construction is that at a late stage s we may see some relatively small α enter s . Now we could twist H_α inside H_{s+1} . But this would cause all the groups H_β for $\beta \in (\alpha, s]$ to be twisted inside H_{s+1} as well. This would result in our twisting at places outside U (and outside F). At the very end this shouldn't matter; we could argue that the differences are washed outside some club of κ . The difficulty though is to explain why *each* group H_γ is free. Naïvely, if t is a limit of such stages s as above, then while $U \cap t$ is nonstationary in t , it is conceivable that the added twisting would cause a stationary amount of twisting, and then $H_{<t}$ would fail to be free. This in fact does not happen, but we prefer to present a modified construction. Our approach is to re-index the filtration. Namely, at stage s we declare that all the groups G_β for $\beta \in (\alpha, s]$ are swallowed inside the new $G_{\alpha+1}$.

So together with the sequence $\langle H_s \rangle$ we define filtrations $\bar{G}_s = \langle G_{\alpha,s} \rangle_{\alpha < s}$ of H_s whose limit will be the desired filtration $\langle G_\alpha \rangle$. So $H_s = G_{s,s}$. The inductive assumption that makes everything work is:

- (i) each H_s is free;
- (ii) $\text{Div}(\bar{G}_s) = s \setminus U_s$.
- (iii) For all $t < s$ and all $\beta < t$, if $U_s \upharpoonright_\beta = U_t \upharpoonright_\beta$ then $G_{\beta,t} = G_{\beta,s}$.

Note that since $U_s \subseteq F$, (ii) implies that for all limit $s < \kappa$, $\text{Div}(\bar{G}_s)$ contains a club of s . Suppose that these objects have been defined for all $t < s$. At stage s we act as follows.

Case 1: s is a successor ordinal. If $U_s = U_{s-1}$ then we let $H_s = H_{s-1} \oplus \mathbb{Z}$, $G_{\alpha,s} = G_{\alpha,s-1}$ for all $\alpha < s$ and $G_{s,s-1} = H_{s-1}$. In this case ensuring that (i) and (ii) above hold for s is immediate.

Suppose that $U_s \neq U_{s-1}$; let α be the least element of $U_s \setminus U_{s-1}$. For all $\beta \leq \alpha$, we let $G_{\beta,s} = G_{\beta,s-1}$. By induction, $G_{\alpha,s} \upharpoonright H_{s-1}$. Write H_{s-1} as the direct sum $G_{\alpha,s} \oplus K_s$. Find a cofinal ω -sequence $\langle \alpha_n \rangle$ in α disjoint from U_s . We let $G_{\alpha+1,s} = \text{twist}(\langle G_{\alpha_n,s} \rangle) \oplus K_s$. So $G_{\alpha+1,s} \supset H_{s-1}$, and $G_{\alpha,s}$ does not detach in $G_{\alpha+1,s}$; but for all $\beta \in \alpha \setminus U_{s-1} = \alpha \setminus U_s$, $G_{\beta,s} \upharpoonright G_{\alpha+1,s}$.

We then go on defining $G_{\beta,s}$ for $\beta \in (\alpha + 1, s)$ as in the previous construction, twisting on elements of U_s and adding copies of \mathbb{Z} outside U_s , taking unions at limit levels; we let $H_s = G_{s,s} = \bigcup_{\beta < s} G_{\beta,s}$. The verification of (i) and (ii) proceeds as in the proof of Theorem 2.16, again using the fact that U_s does not reflect at any $\beta \leq s$. (iii) holds by our instructions.

Case 2: s is a limit ordinal. Let

$$\gamma = \sup \{ \beta < s : \exists t < s (U_t \upharpoonright \beta = U_s \upharpoonright \beta) \}.$$

For all $\beta < \gamma$ we let $G_{\beta,s} = \lim_{t \rightarrow s} G_{\beta,t}$; the limit exists by (iii), and this definition ensures that (iii) holds at s as well. Further, we claim that $H_{<s} = \bigcup_{t < s} H_t$ actually equals $G_{\gamma,s} = \bigcup_{\beta < \gamma} G_{\beta,s}$. This is because for each $t < s$ there is some $r \in (t, s)$ and some $\alpha < \gamma$ which enters U at stage r ; at stage r we define $G_{\alpha+1,r}$ to extend H_t . Now by induction, for all $\beta < \gamma$, $G_{\beta,s}$ is free. We show that (ii) holds: $\text{Div}(\langle G_{\beta,t} \rangle_{\beta < \gamma})$ equals $\gamma \setminus U_s$; this uses the fact that $U_s = \bigcup_{t < s} U_t$. For if $\beta \in U_t$ for some $t < s$ then the construction ensures that for all $r \in [t, s)$, $G_{\beta,r} \not\upharpoonright G_{\beta+1,r}$. And if $\beta \notin U_s$ then for all $t \in (\beta, s)$, for all $\alpha \in (\beta, t)$, $G_{\beta,t} \upharpoonright G_{\alpha,t}$; for each $\alpha \in (\beta, s)$ we can find some $t \in (\beta, s)$ such that $G_{\alpha,t} = G_{\alpha,s}$ and $G_{\beta,t} = G_{\beta,s}$.

Finally, the fact that U_s does not reflect at s implies that $G_{\gamma,s}$ is free. Now as at the successor case, we continue building the sequence $\langle G_{\beta,s} \rangle$ for $\beta \in (\gamma, s)$ (if $\gamma < s$) as in the proof of Theorem 2.16, using U_s , and let $H_s = G_{s,s}$.

This completes the construction; applying the argument above to $s = \kappa$ completes the proof. Also note that for all $\alpha < \kappa$, $|G_\alpha| \leq |\alpha|$, as for each cardinal $\lambda < \kappa$, $U_\lambda = U \upharpoonright \lambda$; this implies that for all $\beta < \lambda$, $G_\beta = G_{\beta,\lambda}$. \square

2.7. The weakly compact case. We turn to the proof of Theorem 1.4(2). Unlike the previous cases, here we not only have to prove hardness, but also membership in the class. This membership follows from an easy characterisation. The boldface version of the following proposition (which applies to weakly compact cardinals) was observed by A. Mekler in his Ph.D. thesis.

Proposition 2.19. *Let κ be an inaccessible cardinal, let G be a group of universe κ , and suppose that κ is $\Pi_1^1(G)$ -indescribable. Then G is free if and only if every subgroup of G of size smaller than κ is free.*

Note that under our assumption that $V = L$, every subgroup of such a group G of size less than κ is an element of L_κ (we say that it is κ -finite). The collection of κ -finite free groups is κ -c.e. (as usual, search for a basis; every basis is κ -finite). This shows that for any $B \in 2^\kappa$, if κ is $\Pi_1^1(B)$ -indescribable then the index-set of the B -computable free groups is $\Pi_2^0(B)$. If κ is weakly compact, this shows that the collection of all free abelian groups of size κ is $\Pi_2^0(L_\kappa)$.

Proof of Proposition 2.19. Recall that the standard filtration of G is defined by letting $G_\alpha = \text{Span}(G \cap \alpha)$, and that we let $\text{Div}(G)$ be the detachment set given by this standard filtration.

Let $\lambda \leq \kappa$ be regular. First note that if λ is closed under the group operation ($G_\lambda = G \upharpoonright_\lambda$) then for all $\alpha < \lambda$, $G_\alpha \in L_\lambda$.

Consider the Σ_1^1 sentence ψ which for such $\lambda \leq \kappa$, says that:

- for all $\alpha < \lambda$, G_α is free (has a basis in L_λ); and
- $\text{Div}(G_\lambda)$ contains a club.

For such λ , $(L_\lambda, G_\lambda) \models \psi$ if and only if G_λ is free. By indescribability, if G is not free then there is some regular $\lambda < \kappa$ such that $G_\lambda = G \upharpoonright_\lambda$ and $(L_\lambda, G \upharpoonright_\lambda) \models \neg\psi$. \square

Assuming that κ is weakly compact, as observed, this implies that the index set of the computable free abelian groups on κ is Π_2^0 . However above the halting problem we can save a quantifier.

Proposition 2.20. *Let κ be weakly compact. Then the collection of free abelian groups on κ is $\Pi_1^0(\emptyset')$ -complete in the collection of groups.*

Proof. First we show that freeness is indeed $\Pi_1^0(\emptyset')$. The point is that if $\lambda < \kappa$ is a cardinal and $H \in L_\lambda$ is a subgroup of G , then H is free if and only if H has a basis in L_λ . So Proposition 2.19 implies that G is free if and only if for all cardinals λ , L_λ sees that every λ -finite subgroup of G is free. The set of cardinals is κ -computable from (indeed κ -equi-computable with) the complete $\Sigma_1^0(L_\kappa)$ set \emptyset' .

For completeness, we first observe that the collection of free abelian groups on κ is Δ_1^0 -hard; this only requires fixing two groups, one free and one not. Now let $A \subseteq 2^\kappa$ be Π_1^0 ; say $Y \in A$ if and only if $(L_\kappa, Y) \models \forall \alpha \psi(\alpha)$, for some formula ψ with bounded quantifiers. Then uniformly in Y we build groups G_α , for $\alpha < \kappa$, such that G_α is free if and only if $(L_\kappa, Y) \models \psi(\alpha)$; and let $G = \bigoplus G_\alpha$. This construction of course relativises to any oracle. \square

The following completes the proof of Theorem 1.4.

Proposition 2.21. *Suppose that κ is inaccessible and Π_1^1 -indescribable. Then the index-set of the computable free abelian groups on κ is Π_2^0 -complete.*

Proof. We have already observed that it is Π_2^0 . We prove hardness. The argument for Proposition 2.20 shows that it is sufficient to prove Σ_1^0 -hardness.

Let A be a κ -c.e. set; we describe a procedure yielding, given $\alpha < \kappa$, a κ -computable group $G = G(\alpha)$ such that $G(\alpha)$ is free if and only if $\alpha \in A$.

The idea is to follow the construction of the proof of theorem Theorem 2.16 up to the next cardinal α^+ (the least $\lambda < \kappa$ such that $\alpha \in L_\lambda$). We twist along E (Definition 2.9) as long as we don't see α enter A . The point is that $\alpha \in A$ if and only if $\alpha \in A_{\alpha^+}$, and that E is stationary in α^+ but not between α and α^+ . So $\alpha \in A$ if and only if $\alpha \in A_{\alpha^+}$ if and only if at some point below α^+ we stop twisting altogether. Once we get to α^+ we cannot continue the construction. Of course, effectively, we don't know that we reached α^+ , so we keep waiting to tell whether it is in E or not; to prevent us from producing a partial group, on the side we keep building a copy of \mathbb{Z}^κ to add to our group.

Here are the details more formally. Fix a κ -effective enumeration $\langle A_s \rangle$ of A ; A_s is the set of $x < s$ such that J_s sees that $x \in A$. For any cardinal λ , $L_\lambda \prec_{\Sigma_1} L_\kappa$, so as promised, for any cardinal λ , $A_\lambda = A \cap \lambda$.

Fix $\alpha < \kappa$. Computably we build an increasing and continuous sequence of groups $\langle H_\beta \rangle_{\beta \in [\alpha, \alpha^+]}$ and a continuous and non-decreasing function $f: [\alpha, \kappa] \rightarrow \alpha^+ + 1$. We then let $G_s = H_{f(s)} \oplus \mathbb{Z}^s$ for all $s \in [\alpha, \kappa]$. This is done so that the sequence $\langle G_s \rangle$ is increasing, continuous and κ -computable, so $G = G_\kappa$ is a κ -computable group. At every stage we increase f by at most one, so for all $t \in [\alpha, \kappa]$, the range of $f \upharpoonright_{[\alpha, t]}$ is an initial segment of $[\alpha, \alpha^+]$; so to define the groups H_β we define the group $H_{f(t)}$ whenever we increase f .

We start with H_α being the trivial group, and $f(\alpha) = \alpha$. Now let $t \in (\alpha, \kappa]$, and suppose that $f(s)$ and $H_{f(s)}$ have been defined for all $s \in [\alpha, t)$. Now there are several options.

Case 1: t is a limit ordinal. We let $f(t) = \sup_{s \in [\alpha, t)} f(s)$. If f is constant on a final segment of t then $H_{f(t)}$ is already defined. Otherwise we let $H_{f(t)} = \bigcup_{s \in [\alpha, t)} H_{f(s)}$.

In the other cases, t is a successor ordinal; let $\beta = f(t - 1)$.

Case 2: β is a successor ordinal. We let $f(t) = \beta + 1$ and $H_{\beta+1} = H_\beta \oplus \mathbb{Z}$.

Case 3: β is a limit ordinal and $t < s(\beta)$. (Of course this includes the case that β is a regular cardinal, which will be α^+ .) We let $f(t) = \beta$.

Case 4: β is a limit ordinal and $t = s(\beta)$. We let $f(t) = \beta + 1$. In this case, by induction, H_β is free; we search for a basis and find it. Also by induction, $\text{Div}(\langle H_\gamma \rangle_{\gamma \in [\alpha, \beta)})$ contains $\gamma \setminus E$, and so contains a club.

If $\beta \in E$ and $\alpha \notin A_t$ then we twist: we find a sequence $\langle \beta_n \rangle$ cofinal in β and disjoint from E , and let $H_{\beta+1} = \text{twist}(\langle H_{\beta_n} \rangle)$.

If $\beta \notin E$, or $\alpha \in A_t$, we let $H_{\beta+1} = H_\beta \oplus \mathbb{Z}$.

This concludes the construction. By induction we can see that $\text{range } f = [\alpha, \alpha^+]$. By induction we see that for all $\beta \in [\alpha, \alpha^+)$, H_β is free, and that if $\alpha \in A$ then $\text{Div}(\langle H_\beta \rangle)$ contains a final segment of α^+ , and otherwise equals $[\alpha, \alpha^+) \setminus E$, which does not contain a club. Hence $\alpha \in A$ if and only if H_{α^+} is free if and only if G is free. \square

Remark 2.22. What about $\Pi_2^0(B)$ -completeness for oracles B which do not compute \emptyset' ? We do not know much, but we can show that if B is low and κ is inaccessible and Π_1^1 -indescribable then the index set of the B -computable free abelian groups is $\Pi_2^0(B)$ -complete in a strong sense: there is a κ -computable (not merely computable in B) function f which reduces the complete $\Pi_2^0(B)$ -set to the set of κ -computable (not just B -computable) free abelian groups.

As above it suffices to prove $\Sigma_1^0(B)$ -hardness. We sketch the argument. Let $\alpha < \kappa$; we effectively build a κ -computable group G which is free if and only if $\alpha \in B'$. Fix a κ -computable approximation $\langle B'_s \rangle_{s < \kappa}$ for B' .

We combine ingredients from previous constructions. We define an increasing and continuous sequence $\langle H_\beta \rangle_{\beta \geq \alpha}$, and along with it an approximation to a filtration $\langle G_\beta \rangle_{\beta \geq \alpha}$ as in the proof of Theorem 2.17. As this is a sketch we ignore this approximation and discuss the final result $\langle G_\beta \rangle$. This is done so that G_β is twisted inside $G_{\beta+1}$ if and only if $\beta \in E$, we see that G_β is free, and $\alpha \notin B'_s$ for all s in some final segment of β . As long as these conditions do not hold we keep ‘‘puffing up’’ the group with copies of \mathbb{Z} so that in the end we do get a κ -computable group; see for example the proof of Theorem 3.10 below.

Why does this work? Suppose first that $\alpha \notin B'$. We will show that for some $\lambda < \kappa$, G_λ is not free. Fix some β such that $\alpha \notin B'_s$ for all $s \geq \beta$, and assume that G_β is free. Then we twist at every $\gamma \in [\beta, \beta^+) \cap E$, which shows that G_{β^+} is not free.

Suppose that $\alpha \in B'$. Since $\beta \in B'_s$ for all s in a final segment of κ , eventually we stop twisting; we just need to show that each G_β is free, that is, the construction does not die prematurely. Since we only twist along E , the first non-free group could only appear at regular cardinal stages $\lambda < \kappa$. Fix such λ . To show (inductively) that G_λ is free, we consider the set C of $\beta < \kappa$ such that cofinally in β we see stages s such that $\alpha \in B'_s$. At no stage $s \in C$ do we twist. The set C is certainly closed, and $C \cap \lambda$ is cofinal in λ because it is κ -computable (with parameter smaller than λ) and C is cofinal in κ .

3. CODING INTO BASES OF FREE GROUPS

Corollary 1.5 says that if κ is a successor cardinal then no reasonable oracle suffices to compute a basis for every computable free abelian group. The situation for inaccessible cardinals remains unclear. In this section we tackle the other direction: what can be coded into all bases of some free abelian group? That is, for which sets $D \in 2^\kappa$ can we find a κ -computable free abelian group, every basis of which κ -computes D ? This is the content of Theorem 1.6, which we prove in this section. In brief, our results say that:

- \emptyset' can always be coded;
- an upper bound on the sets that can be coded is the degree of $\text{Div}(G)$, which is always \emptyset'' -computable, but sometimes \emptyset' -computable;
- in many cases, this upper bound can be realised.

3.1. The limits of coding. Computing bases of a free group is equivalent to computing clubs through the detachment set. The following is an effective version of Proposition 2.6. The proof is the same.

Lemma 3.1. *Let κ be regular and uncountable; let G be a κ -computable free abelian group, and let $\bar{G} = \langle G_\alpha \rangle_{\alpha < \kappa}$ be a κ -computable filtration of G .*

The collection of bases of G and the collection of club subsets of $\text{Div}(\bar{G})$ are κ -Medvedev equivalent. That is, there are partial κ -computable functions $f, g: 2^\kappa \rightarrow 2^\kappa$ such that for every basis B of G , $f(B)$ is a club through $\text{Div}(\bar{G})$; and for every club subset C of $\text{Div}(\bar{G})$, $g(C)$ is a basis of G .

The detachment set is the limit of possible coding into bases.

Theorem 3.2. *Let κ be regular and uncountable, and let G be a κ -computable group. For any $X \in 2^\kappa$ which is not κ -computable from $\text{Div}(G)$, there is a basis of G which does not κ -compute X .*

Proof. The proof of Theorem 3.2 uses effective forcing. It is a generalisation of the forcing notion used to shoot a club through a stationary subset of ω_1 [1]. Of course working effectively we do not actually extend the universe, so we will use the fact that the detachment set does contain a club (as G is free). Fix a regular uncountable cardinal κ and a κ -computable free group G .

The notion of forcing $\mathbb{P} = \mathbb{P}(G)$ we use is the collection of all closed and bounded subsets of $\text{Div}(G)$. The ordering is by end-extension: D extends C if $D \supseteq C$ and $D \cap \max C + 1 = C$. Note that \mathbb{P} is κ -computable from $\text{Div}(G)$.

While \mathbb{P} is not κ -closed, it satisfies a weaker form of closure which will still allow us to build a sufficiently generic filter in κ many steps. It is κ -*strategically closed*. This means that playing against an opponent, we have a strategy to stay inside \mathbb{P} when alternating extending conditions in plays of length $< \kappa$, as long as we get to play at limit stages. In detail, fix a club $D \subseteq \text{Div}(G)$. For $C \in \mathbb{P}$ let $g(C) = C \cup \{\min D \setminus (\max C + 1)\}$. That is, add to C the next element of D beyond $\max C$. If $\gamma < \kappa$ is a limit ordinal and $\langle C_\alpha \rangle_{\alpha < \gamma}$ is a sequence of extending conditions in \mathbb{P} (if $\beta > \alpha$ then C_β extends C_α in \mathbb{P}) such that for any even ordinal $\alpha < \gamma$, $C_{\alpha+1} = g(C_\alpha)$, then letting $C_{<\gamma} = \bigcup_{\alpha < \gamma} C_\alpha$, the condition $C_\gamma = C_{<\gamma} \cup \{\sup C_{<\gamma}\}$ is a condition in \mathbb{P} and extends each C_α . The point of course is that $\sup C_{<\gamma} \in \text{Div}(G)$, as it is in D . In this way we can (within L) build a filter of \mathbb{P} meeting any prescribed collection of κ many dense subsets of \mathbb{P} .

Fix $X \not\leq_\kappa \text{Div}(G)$. Let \mathcal{H} be a filter, sufficiently generic over X ; let $A = \bigcup \mathcal{H}$. This is a closed subset of $\text{Div}(G)$. One kind of dense set we meet ensures that A is unbounded in κ ; we can always extend conditions beyond any point below κ , as $\text{Div}(G)$ is unbounded. It remains to show that $X \not\leq_\kappa A$. The argument is similar to the one used for effective Cohen forcing: if H is 1-generic over Y and Y is noncomputable then $Y \not\leq_T H$; here we need $\text{Div}(G)$ as a base to compute \mathbb{P} . Let Φ be a κ -c.e. functional, and let $C_0 \in \mathbb{P}$. If there is some $C \in \mathbb{P}$ extending C_0 such that $\Phi(C) \perp X$ we take such an extension. Otherwise, we claim that C_0 already forces divergence: there is some $\beta < \kappa$ such that for all $C \in \mathbb{P}$ extending C_0 , $\Phi(C, \beta) \uparrow$. For if not, then using \mathbb{P} (and so using $\text{Div}(G)$) we can κ -compute X by ranging over extensions of C_0 and applying Φ . \square

3.2. Coding \emptyset' . It is not hard to encode \emptyset' . It is possible in all cases, including singular cardinals and ω .

Theorem 3.3. *Let κ be any infinite cardinal. There is a κ -computable free abelian group, every basis of which computes \emptyset' .*

Proof. Begin by constructing a free group on κ generators $\{b_\alpha\}$ for $\alpha < \kappa$. If at stage $s < \kappa$ we see α entering \emptyset' , at that stage we introduce a new generator equal to $b_\alpha/2$

Let B be a basis of the resulting group G . For each α there is a finite subset B_α of B such that $b_\alpha \in \text{Span}(B_\alpha)$, and such a set can be found computably from B ; note that the function $\alpha \mapsto b_\alpha$ is κ -computable. Because B_α is P -independent, $\alpha \in \emptyset'$ if and only if 2 divides b_α in $\text{Span}(B_\alpha)$. Note that even in the case $\kappa = \omega$ determining this is computable, looking at the coefficients of b_α in terms of the generators in B_α . \square

A-priori, for any regular uncountable κ , for any κ -computable group G , $\text{Div}(G)$ is \emptyset'' -computable. Theorem 3.2 shows that if κ is a cardinal for which $\text{Div}(G)$ is \emptyset' -computable for every κ -computable free group G , then Theorem 3.3 is optimal for this κ . In this subsection we outline a number of cases in which this holds.

Proposition 3.4. *Suppose that κ is a regular uncountable cardinal which is not the successor of a non-weakly-compact regular uncountable cardinal. Then for any κ -computable free group G , $\text{Div}(G)$ is \emptyset' -computable.*

Toward finding the complexity of $\text{Div}(G)$, we investigate the complexity of detachment among κ -finite free groups. Fix a regular uncountable cardinal κ . Recall that “ κ -finite” just means “an element of L_κ ”. We first observe that given a κ -finite

group G and a subgroup H , we can effectively find a κ -finite copy of G/H . Using Fact 2.3, this implies:

Lemma 3.5. *The collection of κ -finite free abelian groups is κ -computably equivalent to the collection of κ -finite pairs (G, H) such that G is free and H is a subgroup of G which detaches in G .*

Note that these sets are κ -c.e., and so are \emptyset' -computable. This implies that for any κ -computable free group G , $\text{Div}(G)$ is $\Pi_2^0(L_\kappa)$, and so as promised, \emptyset'' -computable.

To prove Proposition 3.4 we consider several cases, which together cover all cardinals for which the proposition applies:

- (1) κ is the successor of a weakly compact cardinal;
- (2) $\kappa = \omega_1$;
- (3) κ is the successor of a singular cardinal;
- (4) κ is inaccessible.

For cases (1) and (4), the proposition follows immediately from the following:

Proposition 3.6. *Suppose that κ falls under cases (1)–(3). Then the collection of κ -finite free groups is κ -computable.*

Proof. First, we consider cases (1) and (2).

Let H be a κ -finite (torsion free, abelian) group. Let κ^- be the cardinal predecessor of κ . By adding a copy of \mathbb{Z}^{κ^-} we may assume that $|H| = \kappa^-$. Effectively we can find a group G with universe κ^- which is isomorphic to H . Now we use the fact that in both cases, the collection of free groups on κ^- is first-order definable over κ^- . If $\kappa = \omega_1$, then we know that the collection of free groups on ω is Π_3^0 ; by Theorem 1.4, if κ^- is weakly compact, then the collection of free groups on κ^- is $\Pi_2^0(L_\kappa)$. In both cases, whether (L_{κ^-}, G) satisfies this definition can be effectively computed within L_κ .

Next we consider case (3). The proof relies on Shelah's singular compactness theorem [29], see also [7]. Let κ be the successor of a singular cardinal. Shelah's theorem says that (like in the weakly compact case), a group of size κ^- is free if and only if every subgroup of strictly smaller cardinality is free.

Because the collection of κ^- -finite free groups is definable over L_{κ^-} (it is κ^- -c.e.), it is κ -finite. We know that the collection of κ -finite free groups is κ -c.e., so it suffices to show it is also κ -co-c.e. For a κ -finite group G , the collection of all κ -finite subgroups of G cardinality smaller than κ^- is κ -computable (uniformly in G); for each such group H , we can effectively find a κ^- -finite group \hat{H} isomorphic to H , and then see whether it is free or not. \square

We turn to case (4). The following lemma will be also useful later, when we discuss singular cardinals. Recall that if H is a subgroup of a group G then we write $[H, G]$ to denote the collection of all subgroups $K \subseteq G$ such that $H \subseteq K$.

Lemma 3.7. *Let G be a free abelian group, and let H be a subgroup of G . Then $H \mid G$ if and only if $H \mid K$ for all $K \in [H, G]$ such that $|K| = |H|$.*

Proof. Given Proposition 2.4, it suffices to show that if $H \not\mid G$ then there is some $K \in [H, G]$ with $|K| = |H|$ such that $H \not\mid K$. Let B be a basis of G . Let $C \subseteq B$ be a subset of size $|H|$ such that $H \subseteq \text{Span}(C)$; let $K = \text{Span}(C)$. \square

Proof of Proposition 3.4 in the inaccessible case. Suppose that κ is inaccessible. We use Lemma 3.7. We are given a κ -finite subgroup H of G , and want to know whether $H \mid G$ or not. First, we use \mathcal{O}' to find a regular cardinal $\lambda < \kappa$ such that $H \in L_\lambda$. With parameter λ we can computably check, given a κ -finite subgroup $K \in [H, G]$ of size smaller than λ , whether $G_\alpha \mid K$ or not: we search for an isomorphism g from K to a λ -finite group $g[K]$, and ask whether $g[H]$ detaches in $g[K]$; since both $g[K]$ and $g[H]$ are λ -finite, the search for this detachment is performed within L_λ , and so is bounded.

Hence, after finding λ , we can ask \mathcal{O}' the following Σ_1^0 question, equivalent to $H \not\mid G$: is there a κ -finite $K \in [H, G]$ and an injective function g from K into some $\alpha < \lambda$ such that in L_λ , $g[H] \not\mid g[K]$? \square

3.3. Coding \mathcal{O}'' . To finish the proof of Theorem 1.6, we consider the case in which not only $\text{Div}(G)$ can be made to be equivalent to \mathcal{O}'' , but we can code \mathcal{O}'' into all bases of a group; so again in this case our results are tight.

Proposition 3.8. *Suppose that κ is a successor of a regular uncountable cardinal which is not weakly compact. Then the collection of κ -finite free abelian groups is $\Sigma_1^0(L_\kappa)$ -complete.*

Proof. Let κ^- be the cardinal predecessor of κ . The proposition follows from the fact that the collection of free groups with universe κ^- is $\Sigma_1^1(L_{\kappa^-})$ -complete, and that we can effectively translate Σ_1^0 questions about L_κ into Σ_1^1 questions about L_{κ^-} . This is not new but we give the details for completeness. The key is the regularity of $\kappa^- > \omega$, which makes well-foundedness a relatively simple question.

As a first step consider first $\Sigma_1^0(L_\kappa)$ -questions with no parameters. Let φ be a Σ_1^0 formula. To find out if $L_\kappa \models \varphi$, we note that this happens if and only if $L_\alpha \models \varphi$ for some $\alpha < \kappa$. (Actually $\alpha < \omega_1$, but we are doing this step as a warm-up, and this observation won't help later.) Then $L_\kappa \models \varphi$ if and only if there is some $A \subseteq (\kappa^-)^2$ such that (κ^-, A) is a well-founded model of $\text{ZF}^- + (V = L) + \varphi$. Well foundedness is first-order definable in (κ^-, A) , as we only quantify over functions from $\omega \rightarrow \kappa^-$; as κ^- is regular, all of these are κ^- -finite; so this question is $\Sigma_1^1(L_{\kappa^-})$.

Now for the general case, we take a Σ_1^0 formula φ and a parameter $\beta < \kappa$. Effectively, in L_κ , we can find a well-ordering B on κ^- isomorphic to β . Our $\Sigma_1^1(B)$ question now asks for some relation A on κ^- and an embedding of (κ^-, B) into the initial segment of (κ^-, A) determined by some $x \in \kappa^-$ such that $(\kappa^-, A) \models \text{ZF}^- + (V = L) + \varphi(x)$.

To B we can add a fixed $C \subseteq \kappa^-$ such that the collection of free abelian groups is $\Sigma_1^1(C)$ -complete, and so given $\varphi(\beta)$ find a group G on κ^- which is free if and only if $L_\kappa \models \varphi(\beta)$. \square

Corollary 3.9. *Suppose that κ is a successor of a regular cardinal which is not weakly compact. There is a partial κ -computable function which takes as input a κ -finite free abelian group H and a Σ_1^0 formula φ (with parameters in L_κ) and outputs a κ -finite free abelian group $\text{BlackBox}(H, \varphi)$ in which H detaches if and only if φ holds in L_κ .*

Proof. Given H and φ , first use Proposition 3.8 to get a κ -finite group K (of size κ^-) which is free if and only if φ holds in L_κ . Now the idea is to let $G = \text{BlackBox}(H, \varphi)$ be a free extension of H such that $K \cong G/H$, and then refer to Fact 2.3.

Technically what we do is find a surjection f from some copy G of \mathbb{Z}^{κ^-} onto K , ensuring that the kernel of f has size κ^- ; this can be achieved using the freeness of \mathbb{Z}^{κ^-} . Since a subgroup of a free group is free, the kernel of f is isomorphic to \mathbb{Z}^{κ^-} , and so to H . Renaming the elements of G we can thus assume that $H = \ker f$. \square

Theorem 3.10. *Suppose that κ is a successor of a regular cardinal which is not weakly compact. There is a κ -computable free group, all bases of which compute \mathcal{O}'' .*

Proof. We start with a Π_2^0 -complete set P such that for any $X \in 2^\kappa$, if P is X -c.e. then it is X -computable. For example let P be the join of a Π_2^0 -complete set \hat{P} and the collection of all bounded initial segments of \mathcal{O}' : if X enumerates P then it computes \mathcal{O}' , and then \hat{P} is both X -c.e. and X -co-c.e.

Our plan is as follows. For each $\alpha < \kappa$ we fix a κ -finite free group $H(\alpha)$ and produce a κ -computable free group $G(\alpha) \supset H(\alpha)$, such that $H(\alpha)$ detaches in $G(\alpha)$ if and only if $\alpha \in P$. In the construction of $G(\alpha)$ we produce a κ -computable filtration $\langle G_s(\alpha) \rangle_{s < \kappa}$, with $G_0(\alpha) = H(\alpha)$.

Having done that, we let $G = \bigoplus_{\alpha < \kappa} G(\alpha)$ and for $s < \kappa$ we let $G_s = \bigoplus_{\alpha < s} G_s(\alpha)$. Then $\bar{G} = \langle G_s \rangle$ is a filtration of G (which is κ -computable and free). By Lemma 3.1, from any basis of G we effectively obtain a club subset C of $\text{Div}(\bar{G})$. And in fact, from any unbounded subset of $\text{Div}(\bar{G})$ we can enumerate, and hence compute, P . For if $s \in \text{Div}(\bar{G})$ and $\alpha < s$, then $H(\alpha) \mid G(\alpha)$ if and only if $H(\alpha) \mid G_s(\alpha)$: $G_s(\alpha) \mid G_s$ and $G_s \mid G$ so $G_s(\alpha) \mid G(\alpha)$. So we enumerate P from C by enumerating all α such that $H(\alpha) \mid G_s(\alpha)$ for some $s \in C$.

To compute $G(\alpha)$, fix α , and let $H(\alpha)$ be some fixed copy of \mathbb{Z}^{κ^-} . For $s < \kappa$ let $\ell_s = \ell_s(\alpha)$, the “length of witnessing” of $\alpha \in P$, to be

$$\sup \{ \gamma < s : \forall x < \gamma \exists y < s \psi(\alpha, x, y) \},$$

where $\forall x \exists y \psi(\alpha, x, y)$ is a formula defining P in L_κ . So $\langle \ell_s \rangle$ is non-decreasing and continuous, and unbounded in κ if and only if $\alpha \in P$. The idea is the following. Given a length ℓ_s , we extend $G_s(\alpha)$ to potentially twist $H(\alpha)$, to be untangled when we discover a greater length $\ell_t > \ell_s$.

In detail, for each $\beta < \kappa$ let

$$K_\beta = \text{BlackBox}(H(\alpha), “\exists s (\ell_s > \beta)”).$$

That is, K_β is free, $H(\alpha) \subset K_\beta$, and $H(\alpha)$ detaches in K_β if and only if there is some s such that $\ell_s > \beta$.

For each $t < \kappa$, for each $\beta < \ell_t$, we know that $H(\alpha)$ detaches in K_β , and so we can find a complement V_β for $H(\alpha)$ in K_β . We then let

$$G_{t+1}(\alpha) = K_{\ell_t} \oplus \mathbb{Z}^{t+1} \oplus \bigoplus_{\beta < \ell_t} V_\beta.$$

This is done so that $G_{s+1}(\alpha) \subseteq G_{t+1}(\alpha)$ if $s < t$. The idea is that we tangle $H(\alpha)$ in K_{ℓ_s} , until we see that $\ell_t > \ell_s$; we then detach $H(\alpha)$ out of K_{ℓ_s} and tangle it again, this time inside K_{ℓ_t} . This can be thought of as a procedure testing the unboundedness of $\langle \ell_s \rangle$.

Of course, for limit $t \leq \kappa$ we let $G_t(\alpha) = \bigcup_{s < t} G_s(\alpha)$. Note that this can take two forms: if $\langle \ell_s \rangle_{s < t}$ is eventually constant below t , then

$$G_t(\alpha) = K_{\ell_t} \oplus \mathbb{Z}^t \oplus \bigoplus_{\beta < \ell_t} V_\beta;$$

if not, then

$$G_t(\alpha) = H(\alpha) \oplus \mathbb{Z}^t \oplus \bigoplus_{\beta < \ell_t} V_\beta.$$

Note that in both cases, $G_t(\alpha)$ is free. Again this includes the case $t = \kappa$.

If $\alpha \notin P$ then $\langle \ell_s \rangle_{s < \kappa}$ is eventually constant, say with value γ ; in this case, $H(\alpha)$ does not detach in K_γ , and as $K_\gamma \mid G_\kappa(\alpha)$, we see that as required, $H(\alpha) \not\mid G_\kappa(\alpha)$.

If $\alpha \in P$ then $\langle \ell_s \rangle$ does not stabilise below κ , showing that $H(\alpha) \mid G_\kappa(\alpha)$. \square

3.4. More on κ -finite free groups. Propositions 3.4 and 3.6 raise a separate question: in general, what is the complexity of the set of κ -finite free abelian groups? Together with Proposition 3.8, we see that the only case left open is when κ is inaccessible.

Theorem 3.11. *Let κ be inaccessible. Then the collection of κ -finite free abelian groups κ -computes \mathcal{O}' , but does not κ -wtt compute \mathcal{O} (and so is not 1-complete for the class $\Sigma_1^0(L_\kappa)$).*

Proof. Let $\alpha < \kappa$; we want to find out whether $\alpha \in \mathcal{O}'$ or not. We start building an increasing and continuous sequence of groups $\langle G_\beta \rangle_{\beta \in [\alpha, \alpha^+]}$, always twisting at $\beta \in E$. That is, we start with G_α being trivial. We take unions at limit stages. At successors of successors we add a copy of \mathbb{Z} . Suppose that $\beta > \alpha$ is a limit ordinal and G_β is already defined. We consult our oracle to see if G_β is free. If it is, then β will be singular, and so we can wait for $s(\beta)$ and observe if $\beta \in E$ or not; if so we twist G_β inside $G_{\beta+1}$; otherwise we do not. The arguments above show that G_β is free if and only if $\beta < \alpha^+$. So once we see that G_β is not free, we know that $\beta = \alpha^+$, and we can consult L_{α^+} to see whether $\alpha \in \mathcal{O}'$ or not.

Suppose, for a contradiction, that the set of free abelian groups κ -wtt computes \mathcal{O}' ; let Ψ be a reduction. Let $\lambda < \kappa$ be a successor of a singular cardinal, sufficiently large so that the parameter used to compute Ψ is in L_λ . Then the restriction of Ψ to L_λ is in fact a λ -computable reduction of $\mathcal{O}'(L_\lambda)$ to the set of λ -finite abelian groups; this contradicts Proposition 3.6. \square

4. SINGULAR CARDINALS

Recall that even when κ is singular, L_κ is admissible and κ -computability makes sense. When analysing groups with universe κ , though, we need to take care, as the notion of filtration is not as robust. In general, if (L_κ, G) is not admissible, then it is likely that some bounded subsets of G generate subgroups which are unbounded. This does not happen when (L_κ, G) is admissible (for example, when G is κ -computable), as there is a κ -computable function from $B \times \omega$ onto $\text{Span}(B)$. In particular, when G is κ -computable, for any cardinal $\lambda < \kappa$, $G \upharpoonright_\lambda$ is a subgroup of G .

In the absence of well-behaved filtrations we consider the general detachment set, restricted to κ -finite subgroups. Fix a singular cardinal κ and a κ -computable group G . First, for a κ -finite subgroup H of G , let

$$[H, G]_{\text{bdd}} = [H, G] \cap L_\kappa$$

be the collection of κ -finite subgroups of G extending H ; and then let $\text{Div}_{\text{bdd}}(G)$ be the collection of all κ -finite subgroups H of G which detach in every subgroup in $[H, G]_{\text{bdd}}$.

Lemma 4.1. *Suppose that every κ -finite subgroup of G is free. Then the detachment set $\text{Div}_{\text{bdd}}(G)$ is cofinal in $[0, G]_{\text{bdd}}$: every κ -finite subgroup H of G has an extension in $\text{Div}_{\text{bdd}}(G)$.*

Proof. First, note that non-detachment is witnessed at the same cardinality. That is, if H is a κ -finite subgroup of G which is not in $\text{Div}_{\text{bdd}}(G)$, then there is some $K \in [H, G]_{\text{bdd}}$ of size $|H|$ in which H does not detach. To see this simply apply Lemma 3.7 to H and the group $G \upharpoonright_\lambda$, where $\lambda < \kappa$ is regular and sufficiently large to include H , the parameter used for the computable definition of G , and a κ -finite subgroup of G in which H does not detach.

Fix some κ -finite subgroup $H \notin \text{Div}_{\text{bdd}}(G)$; let $\lambda = |H|^+$. Recall that with parameter λ , computing detachment among groups of size $< \lambda$ is κ -computable (see the proof of Proposition 3.4 in the inaccessible case): to tell whether some κ -finite group K detaches in another one P of size $< \lambda$, find a bijection g from P to some $\alpha < \lambda$ and then see if in L_λ we can see a complement for $g[H]$ in $g[P]$.

Suppose, for a contradiction, that H has no extension in $\text{Div}_{\text{bdd}}(G)$. Now we construct a κ -computable filtration $\bar{H} = \langle H_i \rangle_{i < \lambda}$ of a κ -finite group H_λ as follows. Starting with $H_0 = H$, given H_i we find some $H_{i+1} \in [H_i, G]_{\text{bdd}}$ of size $|H| = \lambda^-$ in which H_i does not detach. By the paragraph before, such H_{i+1} can be found κ -effectively. This ensures that for all limit $j \leq \lambda$, the sequence $\bar{H} \upharpoonright_j$ is κ -finite and so $H_j = \bigcup_{i < j} H_i$ is κ -finite. Here again we crucially used the assumption that G is κ -computable.

Now we reached our contradiction: by assumption, H_λ is free. But $\text{Div}(\bar{H})$ is empty, contradicting the fact that it must contain a club of λ (Proposition 2.6). \square

Proposition 4.2. *Suppose that $\text{cf}(\kappa) = \aleph_0$. Then a κ -computable group G is free if and only if every κ -finite subgroup of G is free.*

Again notice that this is stronger than Shelah's singular compactness theorem, as there are many countable subgroups of G which are not κ -finite.

Proof. Suppose that every κ -finite subgroup of G is free. Let $\langle \kappa_n \rangle$ be a cofinal sequence in κ . Define a sequence

$$H_0 \subseteq K_0 \subseteq H_1 \subseteq K_1 \subseteq H_2 \subseteq K_2 \subseteq \dots$$

such that each $K_i \in \text{Div}_{\text{bdd}}(G)$ and $G \upharpoonright_{\kappa_n} \subseteq H_n$; for example we can simply let H_n be the subgroup generated by $K_{n-1} \cup G \upharpoonright_{\kappa_n}$. So $G = \bigcup_n K_n$ and each K_n detaches in K_{n+1} ; the familiar process now gives a basis of G . \square

Corollary 4.3. *Suppose that $X \geq_\kappa \emptyset'$ computes a cofinal ω -sequence in κ . Then every κ -computable free group has an X -computable basis.*

Proof. The sequence $\langle K_n \rangle$ from the proof of Proposition 4.2 is computable from \emptyset' and the sequence $\langle \kappa_n \rangle$, which is \emptyset' -computable; as in Proposition 3.4, $\text{Div}_{\text{bdd}}(G)$ is \emptyset' -computable. \square

So for example, if $\kappa < \aleph_\kappa$ (for example $\kappa = \aleph_\omega$), then every κ -computable free group has a \emptyset' -computable basis: the set of cardinals is \emptyset' -computable, and a cofinal sequence $f: \omega \rightarrow \alpha$ (where $\kappa = \aleph_\alpha$) is κ -finite.

Proposition 4.4. *If $\text{cf}(\kappa) = \aleph_0$, then the index-set of the κ -computable free groups is $\Pi_2^0(L_\kappa)$ -complete.*

Proof. Just like the weakly compact case (Proposition 2.21); the same construction works. \square

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