The Dual Ramsey Theorem and the Property of Baire

Linda Brown Westrick
Joint with Dzhafarov, Flood & Solomon

University of Connecticut, Storrs

February 28th, 2015
South-Eastern Logic Symposium
University of Florida, Gainesville
The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let $[\omega]^k$ denote the set of all $k$-element subsets of $\omega$.

**Theorem (Ramsey’s Theorem)**

If $[\omega]^k = \bigcup_{i<l} C_i$, there is $H \subseteq \omega$ such that $[H]^k \subseteq C_i$ for some $i$. 
The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let $[\omega]^k$ denote the set of all $k$-element subsets of $\omega$.

**Theorem (Ramsey’s Theorem)**

If $[\omega]^k = \bigcup_{i < \ell} C_i$, there is $H \subseteq \omega$ such that $[H]^k \subseteq C_i$ for some $i$.

Instead of $k$-element subsets of $\omega$, we consider partitions of $\omega$ into $k$ pieces.

**Notation:**

If $x \in (\omega)^\omega$ and $y$ is coarser than $x$, we write $y \in (x)^\omega$ (in case $y$ is infinite) or $y \in (x)^k$ (if $y$ has $k$ blocks).
The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let \([\omega]^k\) denote the set of all \(k\)-element subsets of \(\omega\).

**Theorem (Ramsey’s Theorem)**

If \([\omega]^k = \cup_{i<l} C_i\), there is \(H \subseteq \omega\) such that \([H]^k \subseteq C_i\) for some \(i\).

Instead of \(k\)-element subsets of \(\omega\), we consider partitions of \(\omega\) into \(k\) pieces. Notation:

- \((\omega)^k\) is the set of partitions of \(\omega\) into exactly \(k\) pieces.
The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let $[\omega]^k$ denote the set of all $k$-element subsets of $\omega$.

**Theorem (Ramsey’s Theorem)**

If $[\omega]^k = \bigcup_{i<l} C_i$, there is $H \subseteq \omega$ such that $[H]^k \subseteq C_i$ for some $i$.

Instead of $k$-element subsets of $\omega$, we consider partitions of $\omega$ into $k$ pieces. Notation:

- $(\omega)^k$ is the set of partitions of $\omega$ into exactly $k$ pieces.
- $(\omega)^\omega$ is the set of partitions of $\omega$ into infinitely many pieces.
The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let $[\omega]^k$ denote the set of all $k$-element subsets of $\omega$.

**Theorem (Ramsey’s Theorem)**

If $[\omega]^k = \bigcup_{i<l} C_i$, there is $H \subseteq \omega$ such that $[H]^k \subseteq C_i$ for some $i$.

Instead of $k$-element subsets of $\omega$, we consider partitions of $\omega$ into $k$ pieces. Notation:

- $(\omega)^k$ is the set of partitions of $\omega$ into exactly $k$ pieces.
- $(\omega)^\omega$ is the set of partitions of $\omega$ into infinitely many pieces.
- If $x \in (\omega)^\omega$ and $y$ is coarser than $x$, we write $y \in (x)^\omega$ (in case $y$ is infinite) or $y \in (x)^k$ (if $y$ has $k$ blocks.)
The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let \([\omega]^k\) denote the set of all \(k\)-element subsets of \(\omega\).

**Theorem (Ramsey’s Theorem)**

If \([\omega]^k = \cup_{i<l} C_i\), there is \(H \subseteq \omega\) such that \([H]^k \subseteq C_i\) for some \(i\).

Instead of \(k\)-element subsets of \(\omega\), we consider partitions of \(\omega\) into \(k\) pieces. Notation:

- \((\omega)^k\) is the set of partitions of \(\omega\) into exactly \(k\) pieces.
- \((\omega)\omega\) is the set of partitions of \(\omega\) into infinitely many pieces.
- If \(x \in (\omega)\omega\) and \(y\) is coarser than \(x\), we write \(y \in (x)^\omega\) (in case \(y\) is infinite) or \(y \in (x)^k\) (if \(y\) has \(k\) blocks.)

**Theorem (Dual Ramsey Theorem, Carlson & Simpson 1986)**

If \((\omega)^k = \cup_{i<l} C_i\) is Borel, there is \(x \in (\omega)\omega\) such that \((x)^k \subseteq C_i\) for some \(i\).
What’s Known

We write $\text{DRT}_k^l$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:
As usual, applying $\text{DRT}_k^2$ repeatedly yields $\text{DRT}_k^l$.

Open-$\text{DRT}_k^{l+1}$ computably implies $\text{RT}_k^l$. (Miller & Solomon 2004)

For $k \geq 4$, Open-$\text{DRT}_k^l \rightarrow \text{ACA}_0$ over $\text{RCA}_0$. (Miller & Solomon 2004).

Miller & Solomon 2004 and Erhard 2013: various results related to the Carlson-Simpson Lemma, which is the combinatorial core of the DRT.

Our goal: Understand the topological aspects of the DRT.

This is joint work with Damir Dzhafarov, Stephen Flood and Reed Solomon.

Linda Brown Westrick Joint with Dzhafarov, Flood & Solomon (University of Connecticut, Storrs)
What’s Known

We write $DRT^k_l$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.
What’s Known

We write $\text{DRT}_i^k$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:
What’s Known

We write $DRT_i^k$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:
- As usual, applying $DRT_2^k$ repeatedly yields $DRT_i^k$. 

What’s Known

We write $DRT_l^k$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:
- As usual, applying $DRT_2^k$ repeatedly yields $DRT_l^k$.
- Open-$DRT_l^{k+1}$ computably implies $RT_l^k$. (Miller & Solomon 2004)
What’s Known

We write $DRT_l^k$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:

- As usual, applying $DRT_2^k$ repeatedly yields $DRT_l^k$.
- Open-$DRT_l^{k+1}$ computably implies $RT_l^k$. (Miller & Solomon 2004)
- For $k \geq 4$, Open-$DRT_l^k \rightarrow ACA_0$ over $RCA_0$. (Miller & Solomon 2004).
What’s Known

We write $\text{DRT}_l^k$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:

- As usual, applying $\text{DRT}_2^k$ repeatedly yields $\text{DRT}_l^k$.
- Open-\(\text{DRT}_l^{k+1}\) computably implies $\text{RT}_l^k$. (Miller & Solomon 2004)
- For $k \geq 4$, Open-\(\text{DRT}_l^k\) $\rightarrow$ $\text{ACA}_0$ over $\text{RCA}_0$. (Miller & Solomon 2004).
- Miller & Solomon 2004 and Erhard 2013: various results related to the Carlson-Simpson Lemma, which is the combinatorial core of the DRT.
What’s Known

We write $DRT^k_l$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:

- As usual, applying $DRT^k_2$ repeatedly yields $DRT^k_l$.
- Open-$DRT^k_{l+1}$ computably implies $RT^k_l$. (Miller & Solomon 2004)
- For $k \geq 4$, Open-$DRT^k_l \rightarrow ACA_0$ over $RCA_0$. (Miller & Solomon 2004).
- Miller & Solomon 2004 and Erhard 2013: various results related to the Carlson-Simpson Lemma, which is the combinatorial core of the DRT.

Our goal: Understand the topological aspects of the $DRT$. 
What’s Known

We write $DRT_l^k$ for the Dual Ramsey Theorem for $k$ partitions and $l$ colors.

Background knowledge:

- As usual, applying $DRT_2^k$ repeatedly yields $DRT_l^k$.
- Open-$DRT_l^{k+1}$ computably implies $RT_l^k$. (Miller & Solomon 2004)
- For $k \geq 4$, Open-$DRT_l^k$ → $ACA_0$ over $RCA_0$. (Miller & Solomon 2004).
- Miller & Solomon 2004 and Erhard 2013: various results related to the Carlson-Simpson Lemma, which is the combinatorial core of the DRT.

Our goal: Understand the topological aspects of the $DRT$.

This is joint work with Damir Dzhafarov, Stephen Flood and Reed Solomon.
The only effect fancy topology has on $DRT^3+$ is making the comeager approximation to the coloring hard to find.

On the other hand, fancy topology is the only way to give $DRT^2$ content.
The Strength of Topologically Clopen $DRT^{3+}$
The Strength of Topologically Clopen $DRT^{3+}$

Theorem (Dzhafarov, Flood, Solomon, W.)

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.
Theorem (Dzhafarov, Flood, Solomon, W.)

Let \( k \geq 3 \). For each computable ordinal \( \alpha \), there is a \( \emptyset^{(\alpha)} \)-computable clopen coloring of \( (\omega)^k \) such that any homogeneous infinite partition computes \( \emptyset^{(\alpha)} \).

Proof:
The Strength of Topologically Clopen $DRT^{3+}$

**Theorem (Dzhafarov, Flood, Solomon, W.)**

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

**Proof:**

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \bigcup B_i$ is a disjoint union.
The Strength of Topologically Clopen $DRT^3+$

**Theorem (Dzhafarov, Flood, Solomon, W.)**

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

**Proof:**

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \cup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
The Strength of Topologically Clopen $DRT^{3+}$

**Theorem (Dzhafarov, Flood, Solomon, W.)**

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

**Proof:**

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \bigcup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given $\alpha$, let $f$ be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
The Strength of Topologically Clopen $DRT^{3+}$

**Theorem (Dzhafarov, Flood, Solomon, W.)**

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

**Proof:**

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \cup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given $\alpha$, let $f$ be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
- (This means $f \leq_T \emptyset^{(\alpha)}$, and for every $g$ which dominates $f$, $\emptyset^{(\alpha)} \leq_T g$.)
The Strength of Topologically Clopen $DRT^{3+}$

**Theorem (Dzhafarov, Flood, Solomon, W.)**

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

**Proof:**

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \bigcup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given $\alpha$, let $f$ be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
- (This means $f \leq_T \emptyset^{(\alpha)}$, and for every $g$ which dominates $f$, $\emptyset^{(\alpha)} \leq_T g$.)
- Let $p$ be Red if $f(a_p) < b_p$, and Blue otherwise.
The Strength of Topologically Clopen $DRT^{3+}$

Theorem (Dzhafarov, Flood, Solomon, W.)

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

Proof:

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \cup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given $\alpha$, let $f$ be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
- (This means $f \leq_T \emptyset^{(\alpha)}$, and for every $g$ which dominates $f$, $\emptyset^{(\alpha)} \leq_T g$.)
- Let $p$ be Red if $f(a_p) < b_p$, and Blue otherwise.
- Let $x \in (\omega)^\omega$ be an infinite homogeneous partition, $x = \{X_0, X_1, \ldots\}$. 
The Strength of Topologically Clopen $DRT^3+$

Theorem (Dzhafarov, Flood, Solomon, W.)

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

Proof:

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \bigcup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given $\alpha$, let $f$ be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
  
  (This means $f \leq_T \emptyset^{(\alpha)}$, and for every $g$ which dominates $f$, $\emptyset^{(\alpha)} \leq_T g$.)
- Let $p$ be Red if $f(a_p) < b_p$, and Blue otherwise.
- Let $x \in (\omega)^\omega$ be an infinite homogeneous partition, $x = \{X_0, X_1, \ldots\}$.
- Then $x$ is homogeneous for Red; for sufficiently large $M$, consider its coarsening $p = \{X_0, \bigcup_{i=1}^{M-1} X_i, \bigcup_{i=M}^{\infty} X_i\}$
The Strength of Topologically Clopen $DRT^3+$

Theorem (Dzhafarov, Flood, Solomon, W.)

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

Proof:

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \bigcup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given $\alpha$, let $f$ be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
- (This means $f \leq_T \emptyset^{(\alpha)}$, and for every $g$ which dominates $f$, $\emptyset^{(\alpha)} \leq_T g$.)
- Let $p$ be Red if $f(a_p) < b_p$, and Blue otherwise.
- Let $x \in (\omega)^\omega$ be an infinite homogeneous partition, $x = \{X_0, X_1, \ldots\}$.
- Then $x$ is homogeneous for Red; for sufficiently large $M$, consider its coarsening $p = \{X_0, \bigcup_{i=1}^{M-1} X_i, \bigcup_{i=M}^{\infty} X_i\}$
- Then $g(n) := \min X_n$, and $g$ dominates $f$. 

Linda Brown Westrick Joint with Dzhafarov, Flood, Solomon (University of Connecticut, Storrs)
The Strength of Topologically Clopen $DRT^{3+}$

**Theorem (Dzhafarov, Flood, Solomon, W.)**

Let $k \geq 3$. For each computable ordinal $\alpha$, there is a $\emptyset^{(\alpha)}$-computable clopen coloring of $(\omega)^k$ such that any homogeneous infinite partition computes $\emptyset^{(\alpha)}$.

**Proof:**

- For $p \in (\omega)^k$, $p = \{B_0, B_1, B_2, \ldots, B_k\}$, where $\omega = \cup B_i$ is a disjoint union.
- Let $a_p = \min B_1$ and $b_p = \min B_2$. (Note that $\min B_0 = 0$.)
- Given $\alpha$, let $f$ be a self-modulus for $\emptyset^{(\alpha)}$ (Gerdes).
- (This means $f \leq_T \emptyset^{(\alpha)}$, and for every $g$ which dominates $f$, $\emptyset^{(\alpha)} \leq_T g$.)
- Let $p$ be Red if $f(a_p) < b_p$, and Blue otherwise.
- Let $x \in (\omega)^\omega$ be an infinite homogeneous partition, $x = \{X_0, X_1, \ldots\}$.
- Then $x$ is homogeneous for Red; for sufficiently large $M$, consider its coarsening $p = \{X_0, \bigcup_{i=1}^{M-1} X_i, \bigcup_{i=M}^\infty X_i\}$
- Then $g(n) := \min X_n$, and $g$ dominates $f$. 
Criticism of the theorem

This theorem doesn't use the interesting pieces of the DRT. The coloring it produces is topologically clopen. It uses no combinatorics, only growth rate.
Criticism of the theorem

This theorem doesn’t use the interesting pieces of the $DRT$. 
Criticism of the theorem

This theorem doesn’t use the interesting pieces of the $DRT$.

- The coloring it produces is topologically clopen.
This theorem doesn’t use the interesting pieces of the $DRT$.  
- The coloring it produces is topologically clopen.  
- It uses no combinatorics, only growth rate.
Criticism of the theorem

This theorem doesn’t use the interesting pieces of the *DRT*.

- The coloring it produces is topologically clopen.
- It uses no combinatorics, only growth rate.
What this theorem tells us about topology in the $DRT$
What this theorem tells us about topology in the \textit{DRT}

If one wanted to consider topologically interesting Borel colorings of \((\omega)^k\), how would those colorings be represented?

A well-founded Borel code would seem the default. But, a \(\emptyset(\alpha)\)-computable clopen coloring has a computable \(\sim\Delta\alpha\) code.

If we allow well-founded Borel codes to represent topology, the coloring of the previous theorem can’t be avoided. It uses fake topological complexity to hide its \(\Delta\alpha\) information.

In this example, \textit{DRT} \(^3+\) could be seen as a strange way to realize the statement “every Borel set has the property of Baire.”
If one wanted to consider topologically interesting Borel colorings of \((\omega)^k\), how would those colorings be represented?

- A well-founded Borel code would seem the default.
If one wanted to consider topologically interesting Borel colorings of $(\omega)^k$, how would those colorings be represented?

- A well-founded Borel code would seem the default.
- But, a $\emptyset(\alpha)$-computable clopen coloring has a computable $\sim \Delta_\alpha$ code.
What this theorem tells us about topology in the $DRT$

If one wanted to consider topologically interesting Borel colorings of $(\omega)^k$, how would those colorings be represented?

- A well-founded Borel code would seem the default.
- But, a $\emptyset(\alpha)$-computable clopen coloring has a computable $\sim \Delta_\alpha$ code.
- If we allow well-founded Borel codes to represent topology, the coloring of the previous theorem can’t be avoided.
If one wanted to consider topologically interesting Borel colorings of \((\omega)^k\), how would those colorings be represented?

- A well-founded Borel code would seem the default.
- But, a \(\emptyset(\alpha)\)-computable clopen coloring has a computable \(\sim \Delta_\alpha\) code.
- If we allow well-founded Borel codes to represent topology, the coloring of the previous theorem can’t be avoided.
- It uses fake topological complexity to hide its \(\Delta_\alpha\) information.
What this theorem tells us about topology in the $DRT$

If one wanted to consider topologically interesting Borel colorings of $(\omega)^k$, how would those colorings be represented?

- A well-founded Borel code would seem the default.
- But, a $\emptyset^{(\alpha)}$-computable clopen coloring has a computable $\sim \Delta_\alpha$ code.
- If we allow well-founded Borel codes to represent topology, the coloring of the previous theorem can’t be avoided.
- It uses fake topological complexity to hide its $\Delta_\alpha$ information.
- In this example, $DRT^{3+}$ could be seen as a strange way to realize the statement “every Borel set has the property of Baire”
If one wanted to consider topologically interesting Borel colorings of $(\omega)^k$, how would those colorings be represented?

- A well-founded Borel code would seem the default.
- But, a $\emptyset^{(\alpha)}$-computable clopen coloring has a computable $\sim \Delta_\alpha$ code.
- If we allow well-founded Borel codes to represent topology, the coloring of the previous theorem can’t be avoided.
- It uses fake topological complexity to hide its $\Delta_\alpha$ information.
- In this example, $DRT^{3+}$ could be seen as a strange way to realize the statement “every Borel set has the property of Baire”
Carlson and Simpson prove the DRT as follows. Define a variation of $DRT_k$ called $DRT_k^A$. Given an instance of $DRT_k^A$, cook up a set $X$ via $\omega$-many nested applications of various instances of $DRT_{k-1}^A$. Applying the Carlson-Simpson Lemma (combinatorial lemma) to $X$ gives the desired homogeneous partition. As a base case, to solve an instance of $DRT_0^A$, start with a comeager approximation to the given coloring and compute a solution from it.
Carlson and Simpson prove the \textit{DRT} as follows.

Define a variation of \textit{DRT} called \textit{DRT}_k. Given an instance of \textit{DRT}_k, cook up a set \(X\) via \(\omega\)-many nested applications of various instances of \textit{DRT}_{k-1}. Applying the Carlson-Simpson Lemma (combinatorial lemma) to \(X\) gives the desired homogeneous partition.

As a base case, to solve an instance of \textit{DRT}_0, start with a comeager approximation to the given coloring and compute a solution from it.
Carlson and Simpson prove the $DRT$ as follows.
- Define a variation of $DRT^k$ called $DRT_A^k$. 
Carlson and Simpson prove the \( DRT \) as follows.

- Define a variation of \( DRT^k \) called \( DRT^k_A \).
- Given an instance of \( DRT^k \), cook up a set \( X \) via \( \omega \)-many nested applications of various instances of \( DRT^{k-1}_A \).
Carlson and Simpson prove the $DRT$ as follows.

- Define a variation of $DRT^k$ called $DRT_A^k$.
- Given an instance of $DRT^k$, cook up a set $X$ via $\omega$-many nested applications of various instances of $DRT_A^{k-1}$.
- Applying the Carlson-Simpson Lemma (combinatorial lemma) to $X$ gives the desired homogeneous partition.
Carlson and Simpson prove the \textit{DRT} as follows.

- Define a variation of $DRT^k$ called $DRT_A^k$.
- Given an instance of $DRT^k$, cook up a set $X$ via $\omega$-many nested applications of various instances of $DRT_A^{k-1}$.
- Applying the Carlson-Simpson Lemma (combinatorial lemma) to $X$ gives the desired homogeneous partition.
- As a base case, to solve an instance of $DRT_A^0$, start with a comeager approximation to the given coloring and compute a solution from it.
Carlson and Simpson prove the $DRT$ as follows.

- Define a variation of $DRT^k$ called $DRT^k_A$.
- Given an instance of $DRT^k$, cook up a set $X$ via $\omega$-many nested applications of various instances of $DRT^{k-1}_A$.
- Applying the Carlson-Simpson Lemma (combinatorial lemma) to $X$ gives the desired homogeneous partition.
- As a base case, to solve an instance of $DRT^0_A$, start with a comeager approximation to the given coloring and compute a solution from it.
How to prevent coding from masquerading as topology

Idea: Require a $\Delta^\alpha$ coloring to also come equipped with a comeager approximation. (That is, when $(\omega^k)_k = \bigcup_{i<l} C_i$, $C_i$ is $\Delta^\alpha$ insist that along with a $\Delta^\alpha$ code for the $C_i$, one is provided with $\Sigma^1_1$ codes for open sets $U_i$ and $D_n$ such that $\bigcup_{i<l} U_i$ is dense, each $D_n$ is dense and $C_i = U_i$ on $\cap_n D_n$.)

We will see that in fact, the behavior of the coloring on a meager set is irrelevant.
Idea: Require a $\Delta_\alpha$ coloring to also come equipped with a comeager approximation.
How to prevent coding from masquerading as topology

Idea: Require a $\Delta_\alpha$ coloring to also come equipped with a comeager approximation.

(That is, when

$$(\omega)^k = \bigcup_{i<l} C_i, \quad C_i \text{ is } \Delta_\alpha$$

insist that along with a $\Delta_\alpha$ code for the $C_i$, one is provided with $\Sigma_1$ codes for open sets $U_i$ and $D_n$ such that $\bigcup_{i<l} U_i$ is dense, each $D_n$ is dense and

$C_i = U_i \text{ on } \cap_n D_n.$)
How to prevent coding from masquerading as topology

Idea: Require a $\Delta_\alpha$ coloring to also come equipped with a comeager approximation.

(That is, when
\[
(\omega)^k = \bigcup_{i < l} C_i, \quad C_i \text{ is } \Delta_\alpha
\]
insist that along with a $\Delta_\alpha$ code for the $C_i$, one is provided with $\Sigma_1$ codes for open sets $U_i$ and $D_n$ such that $\bigcup_{i < l} U_i$ is dense, each $D_n$ is dense and
\[
C_i = U_i \text{ on } \cap_n D_n.
\]

We will see that in fact, the behavior of the coloring on a meager set is irrelevant.
A coloring of $(\omega)^k$ is reduced if for $p \in (\omega)^k$, the color of $p$ depends only on:

- The least element $a$ of the $k$th block of $p$
- All block membership information for all elements $n < a$.

Reduced colorings are clopen.

**Theorem (DFSW)**

Let $(\omega)^k = \bigcup_{i < l} C_i$ be any coloring that satisfies the property of Baire. Uniformly in a comeager approximation to $\bigcup_{i} C_i$, there is a reduced coloring of $(\omega)^k$ such that any set homogeneous for it computes (together with the comeager approximation) a homogeneous solution to the original.

So, Borel-DRT is reducible to Open-DRT if we rule out coding via the Property of Baire.
An Alternate Proof of the $DRT$

**Definition**

A coloring of $(\omega)^k$ is *reduced* if for $p \in (\omega)^k$, the color of $p$ depends only on:
- The least element $a$ of the $k$th block of $p$
- All block membership information for all elements $n < a$.
An Alternate Proof of the \textit{DRT}

**Definition**

A coloring of $(\omega)^k$ is \textit{reduced} if for $p \in (\omega)^k$, the color of $p$ depends only on:

- The least element $a$ of the $k$th block of $p$
- All block membership information for all elements $n < a$.

Reduced colorings are clopen.
A coloring of \((\omega)^k\) is \textit{reduced} if for \(p \in (\omega)^k\), the color of \(p\) depends only on:
- The least element \(a\) of the \(k\)th block of \(p\)
- All block membership information for all elements \(n < a\).

Reduced colorings are clopen.

**Theorem (DFSW)**

Let \((\omega)^k = \bigcup_{i<l} C_i\) be any coloring that satisfies the property of Baire. Uniformly in a comeager approximation to \(\bigcup_i C_i\), there is a reduced coloring of \((\omega)^k\) such that any set homogeneous for it computes (together with the comeager approximation) a homogeneous solution to the original.
An Alternate Proof of the \textit{DRT}

\textbf{Definition}

A coloring of \((\omega)^k\) is \textit{reduced} if for \(p \in (\omega)^k\), the color of \(p\) depends only on:

- The least element \(a\) of the \(k\)th block of \(p\)
- All block membership information for all elements \(n < a\).

Reduced colorings are clopen.

\textbf{Theorem (DFSW)}

\textit{Let} \((\omega)^k = \bigcup_{i<l} C_i\) \textit{be any coloring that satisfies the property of Baire. Uniformly in a comeager approximation to} \(\bigcup_i C_i\), there is a reduced coloring of \((\omega)^k\) \textit{such that any set homogeneous for it computes (together with the comeager approximation) a homogeneous solution to the original.}

So, Borel-\textit{DRT} is reducible to Open-\textit{DRT} if we rule out coding via the Property of Baire.
An Alternate Proof of the DRT

A coloring of \( (\omega)^k \) is reduced if for \( p \in (\omega)^k \), the color of \( p \) depends only on:

- The least element \( a \) of the \( k \)th block of \( p \)
- All block membership information for all elements \( n < a \).

Let \( k < \omega \) be the set of all finite strings \( \sigma \) on \( \{0, \ldots, k - 1\} \) such that every symbol appears in \( \sigma \) at least once, and the first appearance of \( i \) precedes the first appearance of \( i + 1 \).

The Combinatorial Dual Ramsey Theorem is the DRT for reduced colorings.

**Theorem (Combinatorial Dual Ramsey Theorem (cDRT))**

Let \( (k - 1) < \omega \) be a coloring. Then there is \( x \in (\omega)^\omega \) such that for every \( p \in (x)^k \), \( p \upharpoonright k \in C_i \) for some \( i \), where \( k_p \) is the first element of the \( k \)th block of \( p \).
An Alternate Proof of the DRT

Definition

A coloring of \((\omega)^k\) is *reduced* if for \(p \in (\omega)^k\), the color of \(p\) depends only on:

- The least element \(a\) of the \(k\)th block of \(p\)
- All block membership information for all elements \(n < a\).
An Alternate Proof of the DRT

**Definition**

A coloring of \((\omega)^k\) is *reduced* if for \(p \in (\omega)^k\), the color of \(p\) depends only on:

- The least element \(a\) of the \(k\)th block of \(p\)
- All block membership information for all elements \(n < a\).

Let \(k^{<\omega}_{fin}\) be the set of all finite strings \(\sigma\) on \([0, \ldots, k - 1]\) such that every symbol appears in \(\sigma\) at least once, and the first appearance of \(i\) precedes the first appearance of \(i + 1\).
An Alternate Proof of the \textit{DRT}

**Definition**

A coloring of $(\omega)^k$ is \textit{reduced} if for $p \in (\omega)^k$, the color of $p$ depends only on:
- The least element $a$ of the $k$th block of $p$
- All block membership information for all elements $n < a$.

Let $k_{\text{fin}}^{<\omega}$ be the set of all finite strings $\sigma$ on $\{0, \ldots, k-1\}$ such that every symbol appears in $\sigma$ at least once, and the first appearance of $i$ precedes the first appearance of $i+1$.

The Combinatorial Dual Ramsey Theorem is the \textit{DRT} for reduced colorings.
An Alternate Proof of the \textit{DRT}

\section*{Definition}
A coloring of \((\omega)^k\) is \textit{reduced} if for \(p \in (\omega)^k\), the color of \(p\) depends only on:

- The least element \(a\) of the \(k\)th block of \(p\)
- All block membership information for all elements \(n < a\).

Let \(k_{fin}^<\omega\) be the set of all finite strings \(\sigma\) on \(\{0, \ldots, k - 1\}\) such that every symbol appears in \(\sigma\) at least once, and the first appearance of \(i\) precedes the first appearance of \(i + 1\).

The Combinatorial Dual Ramsey Theorem is the \textit{DRT} for reduced colorings.

\section*{Theorem (Combinatorial Dual Ramsey Theorem (\textit{cDRT}))}
Let \((k - 1)_{fin}^<\omega = \bigcup_{i < l} C_i\) be a coloring. Then there is \(x \in (\omega)^\omega\) such that for every \(p \in (x)^k\), \(p \upharpoonright k_p \in C_i\) for some \(i\), where \(k_p\) is the first element of the \(k\)th block of \(p\).
An Alternate Proof of the DRT

Definition

A coloring of \((\omega)^k\) is reduced if for \(p \in (\omega)^k\), the color of \(p\) depends only on:

- The least element \(a\) of the \(k\)th block of \(p\)
- All block membership information for all elements \(n < a\).

Let \(k_{\text{fin}}^{<\omega}\) be the set of all finite strings \(\sigma\) on \(\{0, \ldots, k - 1\}\) such that every symbol appears in \(\sigma\) at least once, and the first appearance of \(i\) precedes the first appearance of \(i + 1\).

The Combinatorial Dual Ramsey Theorem is the DRT for reduced colorings.

Theorem (Combinatorial Dual Ramsey Theorem (cDRT))

Let \((k - 1)_{\text{fin}}^{<\omega} = \bigcup_{i < l} C_i\) be a coloring. Then there is \(x \in (\omega)^\omega\) such that for every \(p \in (x)^k\), \(p \upharpoonright k_p \in C_i\) for some \(i\), where \(k_p\) is the first element of the \(k\)th block of \(p\).
The Carlson-Simpson Lemma

Let \((k-1)\mathcal{fin} = \bigcup_{i<l} C_i\) be a coloring. Then there is \(x \in (\omega)^\omega\) such that for every \(p \in (x)^k\), \(p \in C_i\) for some \(i\).
The Carlson-Simpson Lemma

**Theorem (Combinatorial Dual Ramsey Theorem (cDRT))**

Let \((k - 1)_{\text{fin}}^\omega = \bigcup_{i<l} C_i\) be a coloring. Then there is \(x \in (\omega)^\omega\) such that for every \(p \in (x)^k\), \(p \in C_i\) for some \(i\).
The Carlson-Simpson Lemma

**Theorem (Combinatorial Dual Ramsey Theorem (cDRT))**

Let \((k - 1)^{<\omega}_{fin} = \bigcup_{i < \ell} C_i\) be a coloring. Then there is \(x \in (\omega)^\omega\) such that for every \(p \in (x)^k\), \(p \in C_i\) for some \(i\).

**Lemma (Carlson-Simpson Lemma)**

Let \((k - 1)^{<\omega}_{fin} = \bigcup_{i < \ell} C_i\) be a coloring. Then there is \(x \in (\omega)^\omega\) such that for every \(p \in (x)^k\) which keeps the first \((k - 1)\) blocks of \(x\) separated, \(p \in C_i\) for some \(i\).
The Carlson-Simpson Lemma

Theorem (Combinatorial Dual Ramsey Theorem ($cDRT'$))

Let $(k-1)_{fin}^{<\omega} = \bigcup_{i<l} C_i$ be a coloring. Then there is $x \in (\omega)^\omega$ such that for every $p \in (x)^k$, $p \in C_i$ for some $i$.

Lemma (Carlson-Simpson Lemma)

Let $(k-1)_{fin}^{<\omega} = \bigcup_{i<l} C_i$ be a coloring. Then there is $x \in (\omega)^\omega$ such that for every $p \in (x)^k$ which keeps the first $(k-1)$ blocks of $x$ separated, $p \in C_i$ for some $i$. 
An Alternate Proof of the \textit{DRT}

Given an instance of \textit{DRT}_k, apply the Property of Baire to get a comeager approximation. Using the comeager approximation, pass to an instance of \textit{cDRT}_k. Define a variation of \textit{cDRT}_k called \textit{CSL}_k (Carlson-Simpson Lemma). Given an instance of \textit{cDRT}_k, cook up a set \(X\) via \(\omega\)-many nested applications of various instances of \textit{CSL}_{k-1}. The result \(X\) is an instance of \textit{cDRT}_{k-1}. The base case is computably true.
An alternate proof of the \textit{DRT}:

Given an instance of \textit{DRT}, apply the Property of Baire to get a comeager approximation. Using the comeager approximation, pass to an instance of $c\textit{DRT}$.

Define a variation of $c\textit{DRT}$ called \textit{CSL} (Carlson-Simpson Lemma). Given an instance of $c\textit{DRT}$, cook up a set $X$ via $\omega$-many nested applications of various instances of $\textit{CSL}^{k-1}$. The result $X$ is an instance of $c\textit{DRT}^{k-1}$.

The base case is computably true.
An alternate proof of the $DRT$:

- Given an instance of $DRT^k$, apply the Property of Baire to get a comeager approximation.
An alternate proof of the *DRT*:

- Given an instance of $DRT^k$, apply the Property of Baire to get a comeager approximation.
- Using the comeager approximation, pass to an instance of $cDRT^k$. 
An alternate proof of the \( DRT \):

- Given an instance of \( DRT^k \), apply the Property of Baire to get a comeager approximation.
- Using the comeager approximation, pass to an instance of \( cDRT^k \).
- Define a variation of \( cDRT^k \) called \( CSL^k \) (Carlson-Simpson Lemma).
An alternate proof of the $DRT$:

- Given an instance of $DRT^k$, apply the Property of Baire to get a comeager approximation.
- Using the comeager approximation, pass to an instance of $cDRT^k$.
- Define a variation of $cDRT^k$ called $CSL^k$ (Carlson-Simpson Lemma).
- Given an instance of $cDRT^k$, cook up a set $X$ via $\omega$-many nested applications of various instances of $CSL^{k-1}$.
An alternate proof of the $DRT$:

- Given an instance of $DRT^k$, apply the Property of Baire to get a comeager approximation.
- Using the comeager approximation, pass to an instance of $cDRT^k$.
- Define a variation of $cDRT^k$ called $CSL^k$ (Carlson-Simpson Lemma).
- Given an instance of $cDRT^k$, cook up a set $X$ via $\omega$-many nested applications of various instances of $CSL^{k-1}$
- The result $X$ is an instance of $cDRT^{k-1}$. 
An alternate proof of the \( DRT \):

- Given an instance of \( DRT^k \), apply the Property of Baire to get a comeager approximation.
- Using the comeager approximation, pass to an instance of \( cDRT^k \).
- Define a variation of \( cDRT^k \) called \( CSL^k \) (Carlson-Simpson Lemma).
- Given an instance of \( cDRT^k \), cook up a set \( X \) via \( \omega \)-many nested applications of various instances of \( CSL^{k-1} \).
- The result \( X \) is an instance of \( cDRT^{k-1} \).
- The base case is computably true.
Implications

Thus, Borel-DRT may be cleanly cleaved into two disparate steps:

Every Borel set has the Property of Baire

Combinatorial Dual Ramsey Theorem

Corollary

The Dual Ramsey Theorem holds for any coloring that has the Property of Baire.

(This possibility was mentioned but not pursued in Carlson & Simpson 1986.)
Thus, Borel-$DRT$ may be cleanly cleaved into two disparate steps:
Thus, Borel-$DRT$ may be cleanly cleaved into two disparate steps:

- Every Borel set has the Property of Baire
Implications

Thus, Borel-$DRT$ may be cleanly cleaved into two disparate steps:
- Every Borel set has the Property of Baire
- Combinatorial Dual Ramsey Theorem
Implications

Thus, Borel-$DRT$ may be cleanly cleaved into two disparate steps:

- Every Borel set has the Property of Baire
- Combinatorial Dual Ramsey Theorem

Corollary

The Dual Ramsey Theorem holds for any coloring that has the Property of Baire.
Thus, Borel-$DRT$ may be cleanly cleaved into two disparate steps:

- Every Borel set has the Property of Baire
- Combinatorial Dual Ramsey Theorem

Corollary

The Dual Ramsey Theorem holds for any coloring that has the Property of Baire.

(This possibility was mentioned but not pursued in Carlson & Simpson 1986.)
Open Questions

How strong is cDRT? (Reverse-math, computable-analysis, descriptive strength.)

Is the Carlson-Simpson Lemma strictly weaker than cDRT?

Linda Brown Westrick Joint with Dzhafarov, Flood & Solomon (University of Connecticut, Storrs)

The Dual Ramsey Theorem and the Property of Baire

February 28th, 2015 South-Eastern Logic Symposium University of Florida, Gainesville 15
Open Questions

How strong is \( cDRT \)? (Reverse-math, computable-analysis, descriptive strength.)
How strong is $cDRT$? (Reverse-math, computable-analysis, descriptive strength.)

Is the Carlson-Simpson Lemma strictly weaker than $cDRT$?
The only effect fancy topology has on $DRT^3+$ is making the comeager approximation to the coloring hard to find.

On the other hand, fancy topology is the only way to give $DRT^2$ content.
The Weakness of $DRT^2$

Theorem (DFSW)

Open-$DRT^2$ is computably true.

Proof: Pass computably to $cDRT^2$. It colors strings of the form $0^n$, so it just colors numbers. Some color is used infinitely often, let’s say Blue. A homogeneous partition is $\{n : 0^n \text{ not Blue}\}$, $\{n_1\}$, $\{n_2\}$, ...

Similarly, if the coloring is given as a comeager approximation, it computes a homogeneous set.
The Weakness of $DRT^2$

**Theorem (DFSW)**

$\text{Open-}DRT^2$ is computably true.
The Weakness of $DRT^2$

**Theorem (DFSW)**

*Open-$DRT^2$ is computably true.*

Proof:

Pass computably to $cDRT^2$. It colors strings of the form $0^n$, so it just colors numbers. Some color is used infinitely often, let’s say Blue. A homogeneous partition is $\{n : 0^n \text{ not Blue}\}$, $\{n_1\}$, $\{n_2\}$, ... Similarly, if the coloring is given as a comeager approximation, it computes a homogeneous set.
The Weakness of $DRT^2$

**Theorem (DFSW)**

*Open-$DRT^2$ is computably true.*

Proof:

- Pass computably to $cDRT^2$. 

---

Linda Brown Westrick Joint with Dzhafarov, Flood & Solomon (University of Connecticut, Storrs)

The Dual Ramsey Theorem and the Property of Baire

February 28th, 2015 South-Eastern Logic Symposium University of Florida, Gainesville
The Weakness of $DRT^2$

**Theorem (DFSW)**

$Open-DRT^2$ is computably true.

**Proof:**

- Pass computably to $cDRT^2$.
- It colors strings of the form $0^n$, so it just colors numbers.
The Weakness of $DRT^2$

**Theorem (DFSW)**

*Open-$DRT^2$ is computably true.*

**Proof:**

- Pass computably to $cDRT^2$.
- It colors strings of the form $0^n$, so it just colors numbers.
- Some color is used infinitely often, let’s say Blue.
The Weakness of $DRT^2$

Theorem (DFSW)

$Open-DRT^2$ is computably true.

Proof:

- Pass computably to $cDRT^2$.
- It colors strings of the form $0^n$, so it just colors numbers.
- Some color is used infinitely often, let’s say Blue.
- A homogeneous partition is $\{n : 0^n \text{ not Blue}\}, \{n_1\}, \{n_2\}, \ldots$. 

Linda Brown Westrick Joint with Dzhafarov, Flood & Solomon (University of Connecticut, Storrs)
The Weakness of $DRT^2$

Theorem (DFSW)

$Open-DRT^2$ is computably true.

Proof:
- Pass computably to $cDRT^2$.
- It colors strings of the form $0^n$, so it just colors numbers.
- Some color is used infinitely often, let’s say Blue.
- A homogeneous partition is $\{n : 0^n \text{ not Blue}\}$, $\{n_1\}$, $\{n_2\}$, ….

Similarly, if the coloring is given as a comeager approximation, it computes a homogeneous set.
The Weakness of $DRT^2$

**Theorem (DFSW)**

$Open-DRT^2$ is computably true.

**Proof:**
- Pass computably to $cDRT^2$.
- It colors strings of the form $0^n$, so it just colors numbers.
- Some color is used infinitely often, let’s say Blue.
- A homogeneous partition is $\{n : 0^n \text{ not Blue}\}, \{n_1\}, \{n_2\}, \ldots$.

Similarly, if the coloring is given as a comeager approximation, it computes a homogeneous set.
The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^3$, it (probably) cannot preserve the computational complexity of its input.

Theorem (DFSW) $cDRT^2$ for $\Delta^0_\alpha$-coded colorings is computably uniformly equivalent to the statement that for every $\Delta^0_\alpha$-coded subset of $\omega$, there is an infinite set contained in either it or its complement.

Proof. Suppose we have a $\Delta^0_\alpha$ subset $A \subseteq \omega$. This could be considered as a 2-coloring for $cDRT^2$. An infinite subset of $A$ or $\overline{A}$ computes a homogeneous partition. An infinite homogeneous partition computes an infinite subset of $A$ or $\overline{A}$.

If the partition is $x = \{X_0, X_1, \ldots\}$, the subset is $\{\min X_1, \min X_2, \ldots\}$.

In general, an infinite subset of $A$ or $\overline{A}$ computes nothing in particular; it could certainly fail to compute $A$. 

Linda Brown Westrick Joint with Dzhafarov, Flood & Solomon (University of Connecticut, Storrs)

The Dual Ramsey Theorem and the Property of Baire

February 28th, 2015 South-Eastern Logic Symposium University of Florida, Gainesville 18
The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^{3+}$, it (probably) cannot preserve the computational complexity of its input.
The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^{3+}$, it (probably) cannot preserve the computational complexity of its input.

**Theorem (DFSW)**

$cDRT^2_2$ for $\Delta_\alpha$-coded colorings is computably uniformly equivalent to the statement that for every $\Delta_\alpha$-coded subset of $\omega$, there is an infinite set contained in either it or its complement.
The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^{3+}$, it (probably) cannot preserve the computational complexity of its input.

**Theorem (DFSW)**

$cDRT^2_2$ for $\Delta_\alpha$-coded colorings is computably uniformly equivalent to the statement that for every $\Delta_\alpha$-coded subset of $\omega$, there is an infinite set contained in either it or its complement.

**Proof.** Suppose we have a $\Delta_\alpha$ subset $A \subseteq \omega$. 

Linda Brown Westrick Joint with Dzhafarov, Flood & Solomon (University of Connecticut, Storrs)
The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^{3+}$, it (probably) cannot preserve the computational complexity of its input.

**Theorem (DFSW)**

$cDRT_2^2$ for $\Delta_\alpha$-coded colorings is computably uniformly equivalent to the statement that for every $\Delta_\alpha$-coded subset of $\omega$, there is an infinite set contained in either it or its complement.

Proof. Suppose we have a $\Delta_\alpha$ subset $A \subseteq \omega$.

- This could be considered as a 2-coloring for $cDRT^2$. 

The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^{3+}$, it (probably) cannot preserve the computational complexity of its input.

**Theorem (DFSW)**

$cDRT^2_2$ for $\Delta_\alpha$-coded colorings is computably uniformly equivalent to the statement that for every $\Delta_\alpha$-coded subset of $\omega$, there is an infinite set contained in either it or its complement.

Proof. Suppose we have a $\Delta_\alpha$ subset $A \subseteq \omega$.

- This could be considered as a 2-coloring for $cDRT^2$.
- An infinite subset of $A$ or $\overline{A}$ computes a homogeneous partition.
The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^{3+}$, it (probably) cannot preserve the computational complexity of its input.

**Theorem (DFSW)**

$cDRT^2_2$ for $\Delta_\alpha$-coded colorings is computably uniformly equivalent to the statement that for every $\Delta_\alpha$-coded subset of $\omega$, there is an infinite set contained in either it or its complement.

Proof. Suppose we have a $\Delta_\alpha$ subset $A \subseteq \omega$.

- This could be considered as a 2-coloring for $cDRT^2$.
- An infinite subset of $A$ or $\overline{A}$ computes a homogeneous partition.
- An infinite homogeneous partition computes an infinite subset of $A$ or $\overline{A}$.

If the partition is $x = \{X_0, X_1, \ldots \}$, the subset is $\{\min X_1, \min X_2, \ldots \}$
The Weakness of $DRT^2$

The principle $DRT^2$ is so weak that unlike $DRT^{3+}$, it (probably) cannot preserve the computational complexity of its input.

**Theorem (DFSW)**

$cDRT^2_2$ for $\Delta_\alpha$-coded colorings is computably uniformly equivalent to the statement that for every $\Delta_\alpha$-coded subset of $\omega$, there is an infinite set contained in either it or its complement.

**Proof.** Suppose we have a $\Delta_\alpha$ subset $A \subseteq \omega$.

- This could be considered as a 2-coloring for $cDRT^2_2$.
- An infinite subset of $A$ or $\overline{A}$ computes a homogeneous partition.
- An infinite homogeneous partition computes an infinite subset of $A$ or $\overline{A}$. If the partition is $x = \{X_0, X_1, \ldots\}$, the subset is $\{\min X_1, \min X_2, \ldots\}$

In general, an infinite subset of $A$ or $\overline{A}$ computes nothing in particular; it could certainly fail to compute $A$. 
Leaving the Property of Baire in $DRT_2^2$

So, using $\Delta^\alpha$ codes for a coloring for $DRT_2^2$ does not make the principle too strong.

We have two related strengthenings of $cDRT_2^2$:

- $cDRT_2^2$ for $\Delta^\alpha$-coded colorings (of $\omega$)
- $DRT_2^2$ for $\Delta^\alpha$-coded colorings (of $(\omega_2^2)$)

Open questions:

Is the second principle strictly stronger than the first?

What if we have a $\Delta^\alpha$ code for a coloring of $(\omega_2^2)$ which we know is clopen?
So, using $\Delta_\alpha$ codes for a coloring for $DRT^2$ does not make the principle too strong.
Leaving the Property of Baire in $DRT^2_2$

So, using $\Delta_\alpha$ codes for a coloring for $DRT^2$ does not make the principle too strong.

We have two related strengthenings of $cDRT^2$:

---

Linda Brown Westrick Joint with Dzhaf: The Dual Ramsey Theorem and the Proj
So, using $\Delta_\alpha$ codes for a coloring for $DRT^2$ does not make the principle too strong.

We have two related strengthenings of $cDRT^2$:

- $cDRT^2$ for $\Delta_\alpha$-coded colorings (of $\omega$)
Leaving the Property of Baire in $DRT_2^2$

So, using $\Delta_\alpha$ codes for a coloring for $DRT^2$ does not make the principle too strong.

We have two related strengthenings of $cDRT^2$:

- $cDRT^2$ for $\Delta_\alpha$-coded colorings (of $\omega$)
- $DRT^2$ for $\Delta_\alpha$-coded colorings (of $(\omega)^2$)
Leaving the Property of Baire in $DRT^2_2$

So, using $\Delta_\alpha$ codes for a coloring for $DRT^2$ does not make the principle too strong.

We have two related strengthenings of $cDRT^2$:

- $cDRT^2$ for $\Delta_\alpha$-coded colorings (of $\omega$)
- $DRT^2$ for $\Delta_\alpha$-coded colorings (of $(\omega)^2$)

Open questions:
Leaving the Property of Baire in $DRT^2_2$

So, using $\Delta_\alpha$ codes for a coloring for $DRT^2$ does not make the principle too strong.

We have two related strengthenings of $cDRT^2$:
- $cDRT^2$ for $\Delta_\alpha$-coded colorings (of $\omega$)
- $DRT^2$ for $\Delta_\alpha$-coded colorings (of $(\omega)^2$)

Open questions:
- Is the second principle strictly stronger than the first?
Leaving the Property of Baire in $\text{DRT}_2^2$

So, using $\Delta_\alpha$ codes for a coloring for $\text{DRT}_2^2$ does not make the principle too strong.

We have two related strengthenings of $\text{cDRT}_2^2$:
- $\text{cDRT}_2^2$ for $\Delta_\alpha$-coded colorings (of $\omega$)
- $\text{DRT}_2^2$ for $\Delta_\alpha$-coded colorings (of $(\omega)^2$)

Open questions:
- Is the second principle strictly stronger than the first?
- What if we have a $\Delta_\alpha$ code for a coloring of $(\omega)^2$ which we know is clopen?
Leaving the Property of Baire in $DRT_2^2$

So, using $\Delta_\alpha$ codes for a coloring for $DRT^2$ does not make the principle too strong.

We have two related strengthenings of $cDRT^2$:

- $cDRT^2$ for $\Delta_\alpha$-coded colorings (of $\omega$)
- $DRT^2$ for $\Delta_\alpha$-coded colorings (of $(\omega)^2$)

Open questions:

- Is the second principle strictly stronger than the first?
- What if we have a $\Delta_\alpha$ code for a coloring of $(\omega)^2$ which we know is clopen?