The Dual Ramsey Theorem and the Property of Baire

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Joint with Dzhafarov, Flood & Solomon

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The Dual Ramsey Theorem

The Dual Ramsey Theorem is a variation of the well-known Ramsey Theorem. Let $[\omega]^k$ denote the set of all k-element subsets of ω.

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If $[\omega]^k = \cup_{i<l} C_i$, there is $H \subseteq \omega$ such that $[H]^k \subseteq C_i$ for some i.

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- $(\omega)^k$ is the set of partitions of ω into exactly k pieces.

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Theorem (Dual Ramsey Theorem, Carlson & Simpson 1986)

If $(\omega)^k = \cup_{i<l} C_i$ is Borel, there is $x \in (\omega)^\omega$ such that $(x)^k \subseteq C_i$ for some i.

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Our goal: Understand the topological aspects of the DRT.

What's Known

We write DRT_l^k for the Dual Ramsey Theorem for k partitions and l colors.

Background knowledge:

- As usual, applying DRT_2^k repeatedly yields DRT_l^k.
- Open-DRT_l^{k+1} computably implies RT_l^k. (Miller & Solomon 2004)
- For $k \geq 4$, Open-$DRT_l^k \to ACA_0$ over RCA_0. (Miller & Solomon 2004).
- Miller & Solomon 2004 and Erhard 2013: various results related to the Carlson-Simpson Lemma, which is the combinatorial core of the DRT.

Our goal: Understand the topological aspects of the DRT.

This is joint work with Damir Dzhafarov, Stephen Flood and Reed Solomon.

The 20 second version of this talk

The only effect fancy topology has on DRT^{3+} is making the comeager approximation to the coloring hard to find.

On the other hand, fancy topology is the only way to give DRT^2 content.

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We will see that in fact, the behavior of the coloring on a meager set is irrelevant.

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So, Borel-DRT is reducible to Open-DRT if we rule out coding via the Property of Baire.

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Theorem (Combinatorial Dual Ramsey Theorem ($cDRT$))

Let $(k-1)^{<\omega}_{fin} = \cup_{i<l} C_i$ be a coloring. Then there is $x \in (\omega)^\omega$ such that for every $p \in (x)^k$, $p \upharpoonright k_p \in C_i$ for some i, where k_p is the first element of the kth block of p.

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Lemma (Carlson-Simpson Lemma)

Let $(k-1)^{<\omega}_{fin} = \cup_{i<l} C_i$ *be a coloring. Then there is* $x \in (\omega)^\omega$ *such that for every* $p \in (x)^k$ **which keeps the first** $(k-1)$ **blocks of** x **separated**, $p \in C_i$ *for some* i*.*

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- The base case is computably true.

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(This possibility was mentioned but not pursued in Carlson & Simpson 1986.)

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Is the Carlson-Simpson Lemma strictly weaker than $cDRT$?

The 20 second version of this talk

The only effect fancy topology has on DRT^{3+} is making the comeager approximation to the coloring hard to find.

On the other hand, fancy topology is the only way to give DRT^2 content.

The Weakness of DRT^2

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In general, an infinite subset of A or $\overline{A}$ computes nothing in particular; it could certainly fail to compute A.

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