Entropy and other subshift invariants

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Subshifts

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- Subshifts are characterized by their set of forbidden strings.
- Different F can define the same subshift.
- Variations: Replace 2 with any finite alphabet. Replace $\mathbb{N}$ with G, where G is $\mathbb{N}^d$ or $\mathbb{Z}^d$ and d is any positive integer. Consider only those subshifts generated by c.e. F, or by finite F.

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- The (d, b)-shift-complex shift X_F where

$$F = \{\sigma : K(\sigma) < d|\sigma| - b\}$$

and K is Kolmogorov complexity. (Miller 2012 showed that for each $d < 1$, there is a b for which this subshift is non-empty; the existence of d-shift complex sequences was first shown by Durand, Levin and Shen 2008.)

- If $X \subseteq A^G$ and $Y \subseteq B^G$ are subshifts, then $X \times Y \subseteq (A \times B)^G$ is a subshift, with the shift operation acting pointwise on each $(x, y) \in X \times Y$.

Computability and Symbolic Dynamics

- Cenzer, Dashti, King 2008 and Cenzer, Dashti, Toska, Wyman 2012. Π^0_1 subshifts, Turing degrees of elements of countable subshifts.

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- Simpson 2011. For any subshift X (including all variations)

$$\text{entropy}(X) = \text{Hausdorf Dimension}(X) = \max_{x \in X} \dim x,$$

where dim is the constructive dimension.

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- Hochman and Meyerovitch 2010. Characterization of the entropies of multidimensional subshifts of finite type.

Outline of the Talk

I will consider three subshift invariants.

- Entropy (expository section)
- Medvedev degree (and its independence from entropy)
- Effective dimension spectrum (to be defined)

Entropy

Entropy

- Definition and properties.
- Characterization of the entropies of Π_1^0 subshifts.
- Characterization of the entropies of shifts of finite type.

Entropy

Definition

For $X \subseteq 2^{\mathbb{N}}$, the entropy of X is

$$\mathrm{ent}(X) = \lim_{n\to\infty} \frac{\log_2 |\{x \upharpoonright n : x \in X\}|}{n}.$$

- This definition generalizes to $X \subseteq 2^G$ for $G = \mathbb{N}^d$ or $\mathbb{Z}^d$; divide by the number of symbols in the sample.
- This definition generalizes to $X \subseteq A^G$ for any finite A. If $|A| > 2$ then the entropy may be greater than 1.

Theorem (Simpson)

For any subshift X, $\mathrm{ent}(X) = \max_{x\in X} \dim x$, *where* $\dim$ *is the effective dimension.*

Entropy - basic properties

Definition

For $X \subseteq 2^{\mathbb{N}}$, the entropy of X is

$$\text{ent}(X) = \lim_{n\to\infty} \frac{\log_2 |\{x \upharpoonright n : x \in X\}|}{n}.$$

- If X and Y are subshifts,

$$\text{ent}(X \times Y) = \text{ent}(X) + \text{ent}(Y).$$

- If X is a Π^0_1 subshift, $\text{ent}(X)$ is right-r.e.

Entropy characterization, Π^0_1 case

Fact (Folklore?, Hertling-Spandl 2008)

For any (right-r.e.) $s \in [0,1)$, there is a one-dimensional (Π^0_1) subshift with entropy s.

A proof:

- Letting $s = 0.b_1 b_2 b_3 \ldots$ be the binary expansion of s, define a sequence $t \in \{0, *\}^{\mathbb{N}}$ as follows.
- If $b_1 = 0$, set every other bit of t to 0; else, set every other bit of t to $*$.
- Repeat for $b_2, b_3, \ldots$, each time filling in half the remaining bits.
- The density of $*$ in t is s.
- Forbid σ if it is impossible to replace some bits of σ with $*$ and obtain a subword of t.

Fact (Folklore?)

In the previous Fact, "one-dimensional" may be replaced by "n-dimensional" for any n.

Entropy characterization, shifts of finite type

Definition

A subshift X is a *shift of finite type* (SFT) if X is obtained by forbidding finitely many words.

Fact (classical)

The entropies of the one-dimensional SFTs are exactly the rational multiples of the logarithms of the spectral radii of matrices with positive integer coefficients. (Perron-Frobenius theory)

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Theorem (Hochman-Meyerovitch 2010)

For $n > 1$, the entropies of the n-dimensional SFTs are exactly the right-r.e. numbers.

Medvedev Degree
and its independence from entropy

Medvedev Degree

and its independence from entropy

- Definition and characterization.
- Independence of Medvedev degree and entropy.
- Independence in 2^G.
- A corollary concerning entropy.

Medvedev degree

Definition

Give $\mathcal{A}, \mathcal{B} \subseteq 2^{\mathbb{N}}$, we say $\mathcal{A}$ is Medvedev reducible to $\mathcal{B}$ if there is a Turing functional Γ such that for all $X \in \mathcal{B}$, $\Gamma(X) \in \mathcal{A}$.

Of course this generalizes when $2^{\mathbb{N}}$ is replaced with A^G.

Theorem (Simpson 2007)

Let P be any Π^0_1 class. Then there is a two-dimensional SFT Medvedev equivalent to P.

Theorem (J. Miller 2012)

Let P be any Π^0_1 class. Then there is a one-dimensional Π^0_1 subshift Medvedev equivalent to P.

Independence of entropy and Medvedev degree

Proposition

For any right-r.e. s and any Π^0_1 set P, there is a one-dimensional Π^0_1 subshift with entropy s that is Medvedev equivalent to P.

Proof:

- Let M_P be a Π^0_1 subshift Medvedev equivalent to P as constructed by Miller.
- Let X_s be a Π^0_1 subshift with entropy s as constructed earlier.
- One may verify that $\text{ent}(M_P) = 0$, so $\text{ent}(M_P \times X_s) = \text{ent}(X_s) = s$.
- One may verify that $M_P \times X_s$ is Medvedev equivalent to P.

Proposition

In the previous proposition, one may replace "one-dimensional Π^0_1 subshift" with "two-dimensional SFT".

Proof: The same, but using the constructions of Simpson 2007 and Hochman & Meyerovitch 2010.

Independence in 2^G

The results of the previous slide used a symbol set of size at least four. In order to obtain subshifts on only two symbols, a different strategy is needed.

Theorem

For any Π^0_1 set P and any right-r.e. $s \in [0,1)$, there is a subshift $X \subseteq 2^{\mathbb{N}}$ Medvedev equivalent to P with entropy s.

Theorem

In the previous theorem, "a subshift $X \subseteq 2^{\mathbb{N}}$" may be replaced with "a SFT $X \subseteq 2^{\mathbb{Z}^2}$".

Independence of Medvedev degree and entropy in $2^{\mathbb{N}}$

Theorem

For any Π^0_1 set P and any right-r.e. $s \in [0,1)$, there is a subshift $X \subseteq 2^{\mathbb{N}}$ Medvedev equivalent to P with entropy s.

Proof:

- Let N be odd and large enough that $s' = \frac{N}{N-1}s < 1$.
- Let $X_{s'}$ be a subshift of entropy s' as described previously.
- Let M_P be a certain subshift of entropy zero and Medvedev degree of P.
- The desired subshift $Z \subseteq 2^{\mathbb{N}}$ is defined so that its elements z are exactly those with the following property: there exist $x \in X_{s'}$ and $m \in M_P$ so that $z = x$ on density $\frac{N-1}{N}$ of its bits, and on the remaining density $\frac{1}{N}$ bits (evenly spaced), the bits of m are recorded.
- The effect of the superposition of the bits of m is to erase $\frac{1}{N}$ of the information from x, so the entropy of Z is s.
- One may verify the Medvedev degree.

Independence of Medvedev degree and entropy in $2^{\mathbb{Z}^2}$

Theorem

For any Π^0_1 set P and any right-r.e. $s \in [0,1)$, there is a SFT $X \subseteq 2^{\mathbb{Z}^2}$ Medvedev equivalent to P with entropy s.

- From Simpson 2007, we have a 2-dimensional SFT M_P with the right Medvedev degree.
- From Hochman & Meyerovitch 2010, we can produce a 2-dimensional SFT X with any right-r.e. entropy.
- Both the above use many symbols! Each element of M_P, X, encodes a run of a Turing machine.
- Each symbol of $x \in X$ optionally encodes one bit of information; the computation encoded by x determines the information density, and this density is capped at the desired entropy.

Independence of Medvedev degree and entropy in $2^{\mathbb{Z}^2}$

Theorem

For any Π^0_1 set P and any right-r.e. $s \in [0,1)$, there is a SFT $X \subseteq 2^{\mathbb{Z}^2}$ Medvedev equivalent to P with entropy s.

Proof:

- Build an SFT Z out of M_P and a HM-like X as follows.
- Reserve a large density of bits of each $z \in Z$ for entropy, leaving the rest for computation.
- In the computation portion, represent the symbols of M_P and X using a QR-code style representation.
- Instead of each symbol of X controlling the expression of one bit of information, let each bit of X control the expression of many bits from the nearby entropy part of the subshift.
- Adjust the density of the entropy part and the density of coding bits of X so that they cancel each other, resulting in the desired entropy for Z.

Effective dimension spectrum

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- Definition and properties
- Examples
- Minimal subshifts
- Open questions

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Definition

Let X be a subshift. The *effective dimension spectrum* of X is

$$\mathcal{DS}(X) = \{\dim(x) : x \in X\}.$$

Question: What sets $A \subseteq [0,1]$ can be effective dimension spectra?

Restrictions on $\mathcal{DS}(X)$

- $\mathcal{DS}(X)$ is $\boldsymbol{\Sigma^1_1}$.

$$s \in \mathcal{DS}(X) \iff \exists x \in X \liminf_{n\to\infty} \frac{K(x \upharpoonright n)}{n} = s.$$

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- By the previously quoted theorem of Simpson, $\max_{x \in X} \dim x$ exists. So $\mathcal{DS}(X)$ has a maximum element.

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- By the previously quoted theorem of Simpson, $\max_{x \in X} \dim x$ exists. So $\mathcal{DS}(X)$ has a maximum element.
- If σ does not appear in any sequence of X, then all sequences of X may be compressed by some fixed fraction related to the length of σ. Therefore, unless $X = 2^{\mathbb{N}}$, $\mathcal{DS}(X)$ is bounded away from 1.

Examples of Dimension Spectra

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A shift of finite type is a shift of the form X_F where F is finite.

Theorem

If X is a shift of finite type, then $\mathcal{DS}(X) = [0, \text{entropy}(X)]$.

Most commonly encountered subshifts have dimension spectrum of the form $[0, \text{ent}\, X]$.

Initial Interval Spectra

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Proof:

- We have $0, \mathrm{ent}(X) \in \mathcal{DS}(X)$ by computable extendibility and Simpson's theorem.
- Given $s \in (0, \mathrm{ent}\, X)$, build x by finite extension. Following Durand-Levin-Shen and Hirschfeldt-Kach, increase the length by the same amount each time.
- If $\frac{K(x \upharpoonright n)}{n} > s$, choose the extension computably to decrease the information density.
- Otherwise, by uniform fullness there is a way to extend which increases the information density.

The Shift Complex Sequence Family

So far every example we have seen has had $\mathcal{DS}(X) = [0, \mathrm{ent}(X)]$. But there are spectra which are bounded away from zero.

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So far every example we have seen has had $\mathcal{DS}(X) = [0, \mathrm{ent}(X)]$. But there are spectra which are bounded away from zero. The (d, b)-shift-complex subshift X_F, where $F = \{\sigma : K(\sigma) < d|\sigma| - b\}$ satisfies

$$d \in \mathcal{DS}(X_F) \subseteq [d, \mathrm{entropy}(X)]$$

but I do not know if the containment is proper.

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- Let X be a shift wth dimension spectrum $[0, \frac{1}{3}]$. (Such X exists.)
- Let Y be a $(\frac{2}{3}, b)$-shift-complex subshift.
- Then there is a gap: $\frac{1}{3}, \frac{2}{3} \in \mathcal{DS}(X \cup Y)$, but $(\frac{1}{3}, \frac{2}{3}) \notin \mathcal{DS}(X \cup Y)$.

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Restricting attention to minimal subshifts rules out the silly example of the previous slide. However, some sequences do not belong to any minimal subshift, so just understanding minimal subshifts will not be enough.

Minimal Subshifts

Proposition

If X is minimal, $\mathcal{DS}(X)$ has a least element.

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Proof.

- Going in stages, construct X by finite extension, $\sigma_0 \prec \sigma_1 \prec \cdots$
- We'll maintain always that σ_n is a subword of some $y \in X$, and thus by minimality, that σ_n is a subword of every $y \in X$.
- At stage $n+1$, find $y \in X$ so that $\dim(y) - \inf \mathcal{DS}(X) < 1/n$.
- Let $\sigma_{n+1} = y[k, k+m]$, where k is the start of σ_n in y, and m is long enough that

$$\frac{K(\sigma_{n+1})}{|\sigma_{n+1}|} - \dim(y) < 1/n.$$

- Then

$$\frac{K(\sigma_{n+1})}{|\sigma_{n+1}|} - \inf \mathcal{DS}(X) < 2/n.$$

□

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- Let $\mathcal{E}_1$ be a collection of n_1 strings of length l_1.
- For each i, let $\mathcal{E}_i$ be the set of strings made from concatenating all the strings of $\mathcal{E}_{i-1}$ in all possible orders. (So $n_i := |\mathcal{E}_i| = n_{i-1}!$, and $l_i := |\sigma|$ for $\sigma \in \mathcal{E}_i$ satisfies $l_i = n_{i-1} l_{i-1}$.)
- Let M be the subshift whose permitted words are exactly the subwords of strings from $\cup_i \mathcal{E}_i$.

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Facts

- M is minimal. (Every word of $\mathcal{E}_i$ appears in every word of length $2l_{i+1}$.)
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- For appropriate choices of n_1 and l_1, M has positive entropy.

Theorem

The subshift M has dimension spectrum $[0, \mathrm{ent}(M)]$.

Questions

- What sets can be $\mathcal{DS}(X)$?
- Is $\mathcal{DS}(X)$ closed?
- If not, does it always have a minimum element?
- If $\mathcal{DS}(X)$ is bounded away from zero, must X contain shift-complex sequences?
- How do these answers change if X is minimal?
- If the spectra of minimal subshifts were well-characterized, how would one use that to explore the spectra of arbitrary subshifts?
- What is the dimension spectrum of the shift-complex subshift?
- Are there any natural examples of subshifts whose effective spectra are bounded away from 0?
- What is the relationship between effective dimension spectrum and Medvedev degree?