AN EFFECTIVE ANALYSIS OF THE DENJOY RANK

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Abstract. We analyze the descriptive complexity of several Π^1_1 ranks from classical analysis which are associated to Denjoy integration. We show that VBG, VBG_*, ACG and ACG_* are Π^1_1-complete, answering a question of Walsh in case of ACG_*. Furthermore, we identify the precise descriptive complexity of the set of functions obtainable with at most α steps of the transfinite process of Denjoy totalization: if $|\cdot|$ is the Π^1_1-rank naturally associated to VBG, VBG_* or ACG_*, and if $\alpha < \omega_1^{ck}$, then $\{F \in C(I) : |F| \leq \alpha\}$ is $\Sigma^0_{2\alpha}$-complete. These finer results are an application of the author's previous work on the limsup rank on well-founded trees. Finally, $\{(f, F) \in M(I) \times C(I) : F \in ACG_* \text{ and } F' = f \text{ a.e.}\}$ and $\{f \in M(I) : f \text{ is Denjoy integrable}\}$ are Π^1_1-complete, answering more questions of Walsh.

Real analysis of the early 20th century featured a number of naturally occuring Π^1_1-complete sets. The most prominent example may be the set of differentiable functions on the unit interval I, considered as a subset of $C(I)$, the metric space of continuous functions on I with the supremum norm.

Any Π^1_1 set A may be decomposed as a transfinite union $\cup_{\alpha<\omega_1} A_\alpha$, where each A_α is Borel. When this is done in a sufficiently uniform way, the function which maps each $f \in A$ to the least α such that $f \in A_\alpha$ is called a Π^1_1-rank. Sometimes a Π^1_1 set has an obvious and natural Π^1_1-rank. This is the case for the set of well-founded trees $T \subseteq \omega^{<\omega}$ (the usual well-founded tree rank), the collection of countable closed subsets of I (the Cantor-Bendixson rank), and the collection of continuous functions obtainable by Denjoy totalization, as we shall see shortly. Other times, finding a rank that could be considered natural is more difficult. For example, in [KW86] Kechris and Woodin defined a suitably natural rank on the set of differentiable functions.

When the rank on A is natural, it becomes meaningful to ask for the precise descriptive complexity of the initial segments A_α. This was done implicitly for the well-founded tree rank in [GMS13], for the Cantor-Bendixson rank in [Lem87], and for the Kechris-Woodin rank on differentiable functions in [Wes14]. The purpose of this paper is to show that the method used in [Wes14] generalizes to give descriptive complexities for three hierarchies from classical analysis related to Denjoy totalization. We also give a new proof of Lempp's result on the descriptive complexity of the initial segments of the Cantor-Bendixson rank.

Denjoy totalization is a transfinite integration process developed by Denjoy in 1912 to solve the problem of recovering F from F' whenever F is an everywhere differentiable function in $C(I)$. The process does a little more, recovering also some a.e. differentiable F, but not all of them. The set of $F \in C(I)$ recoverable from F' by Denjoy totalization is denoted ACG_*. The related sets ACG, VBG_* and VBG are described in the next section.

The main result is the following. Let $|\cdot|_{VB}$, $|\cdot|_{VB_*}$, $|\cdot|_{AC}$ and $|\cdot|_{AC_*}$ denote the natural Π^1_1 ranks on VBG, VBG_*, ACG and ACG_* respectively.

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Theorem 1. *Let $X = VB, VB_*$ or AC_*, let $Y \in 2^\omega$, let $1 < \alpha < \omega_1^Y$, and let*

$$A_\alpha = \{F \in C(I) : |F|_X \leq \alpha\}.$$

Then A_α is $\Sigma^0_{2\alpha}(Y)$, and for any $\Sigma^0_{2\alpha}(Y)$ set B, there is a Y-computable reduction from B to A_α. In particular, A_α is $\boldsymbol{\Sigma}^0_{2\alpha}$-complete, and if $\alpha < \omega_1^{CK}$, then A_α is $\Sigma^0_{2\alpha}$-complete.

Let $M(I)$ denote the Polish space of measurable functions on the unit interval, with metric given by $d(f,g) = \int_I \min(1, |f-g|)$. Our next theorem answers three questions in [Wal17].

Theorem 2. *The following sets are all Π^1_1-complete:*

1. *VBG, VBG_*, ACG and ACG_**
2. *$\{(f,F) \in M(I) \times C(I) : F \in ACG_*$ and $F' = f$ a.e.$\}$*
3. *$\{f \in M(I) : F$ is Denjoy integrable$\}$*

The sets $\{F \in C(I) : |F|_X \leq 1\}$ are better known as the collection of functions of bounded variation when $X = VB, VB_*$ and the collection of absolutely continuous functions when $X = AC, AC_*$. The following results filling in the $\alpha = 1$ case of Theorem 1 have routine proofs but their statements may be of interest.

Theorem 3. (1) *The set of continuous functions of bounded variation is Σ^0_2-complete.*
(2) *The set of absolutely continuous functions is Π^0_3-complete.*

In Section 1, after some preliminaries we review the main tool in [Wes14], an analysis of the descriptive complexity of the limsup rank on the set of well-founded trees. We also give the background on Denjoy totalization and the hierarchies ACG_*, ACG, VBG_* and VBG. In Section 2 we apply this analysis to the simpler case of the Cantor-Bendixson derivative, recovering the aforementioned result of Lempp. In Section 3 we prove hardness results for the hierarchies on ACG_*, ACG, VBG_* and VBG, obtaining Theorem 2 and one direction of Theorem 1. In Section 4 we obtain matching descriptive results for all of these hierarchies except ACG, giving the other direction of Theorem 1. Section 5 contains open questions.

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1. Preliminaries

1.1. **Notation.** We use standard computability-theoretic notation. Trees $T \subseteq \omega^{<\omega}$ can be encoded by elements of Cantor space. Let $Tr \subseteq 2^\omega$ be the set of codes for such trees. Given $T \in Tr$, we usually forget the encoding and treat T as a subset of $\omega^{<\omega}$ anyway. The set $[T] \subseteq \omega^\omega$ is the set of paths through T. Let $\sigma^\frown\tau$ denote the concatenation of σ and τ. If $\sigma \in T$, let T_σ denote $\{\tau : \sigma^\frown\tau \in T\}$, and for any $\sigma \in \omega^{<\omega}$, let $\sigma^\frown T$ denote $\{\sigma^\frown\tau : \tau \in T\}$. Let $[\sigma]$ denote $\{X \in \omega^\omega : \sigma \prec X\}$ or $\{X \in 2^\omega : \sigma \prec X\}$, depending on the context. When a tree is in fact a subset of $2^{<\omega}$, we usually give it the name S, reserving T for trees in Baire space.

The complement of a set $A \subseteq \omega^\omega$ is denoted $\neg A$. If $A, B, C, D \subseteq \omega^\omega$ and f is a computable function from $A \cup B$ to $C \cup D$ with $f^{-1}(C) = A$ and $f^{-1}(D) = B$, we say that f is a computable reduction from (A, B) to (C, D). If $B = \neg A$ and $D = \neg C$, we say f is a computable reduction from A to C.

A tree $T \subseteq \omega^{<\omega}$ is well-founded, denoted $T \in WF$, if $[T] = \emptyset$. A set $A \subseteq \omega^\omega$ is Π^1_1 if there is a computable reduction from A to WF. A Π^1_1 set A is Π^1_1-complete if there is a computable function from WF to A. The boldface notions $\mathbf{\Pi}^1_1$ and $\mathbf{\Pi}^1_1$-complete are obtained by replacing "computable" with "continuous", or equivalently, relativizing to an oracle.

For $n < \omega$, a set $X \in 2^\omega$ is Σ^0_n-complete if $X \equiv_1 \emptyset^{(n)}$. To extend the notion of completeness to the ordinals, we assume some familiarity with hyperarithmetic theory: Kleene's $\mathcal{O}$, ordinal notations, the sets H_a for $a \in \mathcal{O}$. For $a \in \mathcal{O}$, we use $|a|_{\mathcal{O}}$ to denote the ordinal coded by a. The supremum of ordinals represented in $\mathcal{O}$ is ω_1^{CK}. Following [AK00], if $n \geq \omega$ we say that a subset $X \subseteq \omega$ is Σ^0_α-complete if $X \equiv_1 H_{2^a}$, where $a \in \mathcal{O}$ is any notation with $|a|_{\mathcal{O}} = \alpha$. All these notions can relativize to an oracle $Y \in 2^\omega$, using notation $\mathcal{O}^Y$, $|a|^Y_{\mathcal{O}}$, H^Y_a, ω^Y_1, $\Sigma^0_\alpha(Y)$.

These computability notions on subsets of ω can be extended to effective topological notions on subsets of ω^ω. A set $A \subseteq \omega^\omega$ is $\Sigma^0_\alpha(Y)$ if $\omega \leq \alpha < \omega^Y_1$ and there is an index e_0 and a notation $a \in \mathcal{O}^{Y\oplus\emptyset}$ such that for all $X \in \omega^\omega$, $a \in \mathcal{O}^{Y\oplus X}$ with $|a|^{Y\oplus X} = \alpha$ and

$$X \in A \iff e_0 \in H^{Y\oplus X}_{2^a}.$$

If $\alpha < \omega$, replace $H^{Y\oplus X}_{2^a}$ with $H^{Y\oplus X}_a$. When showing that a set is $\Sigma^0_\alpha(Y)$, we usually use effective transfinite recursion. We assume familiarity with this method and sometimes avoid explicit mention of the sets $H^{Y\oplus X}_{2^a}$ while using it.

It is well-known that $\mathbf{\Sigma}^0_\alpha = \bigcup_{\substack{Y\in 2^\omega \\ \alpha<\omega_1^Y}} \Sigma^0_\alpha(Y)$, where $\mathbf{\Sigma}^0_\alpha$ refers to the αth level of the Borel hierarchy.

We use I to refer to the unit interval, $C(I)$ for the space of continuous real-valued functions on I with the supremum norm, and $M(I)$ for the space of measurable functions on the unit interval, with metric given by $d(f, g) = \int_I \min(1, |f - g|)$. In $C(I)$, the piecewise linear functions with rational endpoints form a countable dense subset; computing a element F of $C(I)$ means providing, uniformly in ε, a piecewise linear function within ε of F in the supremum norm. In $M(I)$, the step functions with rational values and discontinuities at finitely many rational points form a countable dense subset, and can be used as the ideal points in a representation of $M(I)$ as an effectively presented metric space. Thus computing an element f of $M(I)$ means providing, uniformly in ε, an ideal point of the type described above which is ε-close to f.

1.2. **Limsup rank.** Our main tool is the *limsup rank* on well-founded trees, which was defined in [Wes14] as follows.

Definition 1. *Let $T \subseteq \omega^{<\omega}$ be a well-founded tree. If $T = \emptyset$, define $|T|_{ls} = 0$. Otherwise, define*

$$|T|_{ls} = \max(\sup_n |T_n|_{ls}, (\limsup_n |T_n|_{ls}) + 1).$$

This rank is designed to line up nicely with Cantor-Bendixson type derivation processes in a way that will be explained below.

Theorem 4 ([Wes14])**.** *For all constructive $\alpha > 1$, $\{e : \phi_e$ codes a tree T with $|T|_{ls} \leq \alpha\}$ is $\Sigma_{2\alpha}$-complete.*

The only reason for not allowing $\alpha = 1$ in the above theorem is that it is Π_2 to tell whether ϕ_e is total, that is, whether it codes anything at all.

Almost for free, we can make topological claims in addition to computational ones. The next theorem follows from Theorem 4 by relativization.

Theorem 5. *For any nonzero $\alpha < \omega_1$, let*

$$A_\alpha = \{T \in Tr : |T|_{ls} \leq \alpha\}.$$

Then if $\alpha < \omega_1^Y$, we have $A_\alpha \in \Sigma^0_{2\alpha}(Y)$, and for any set $B \in \Sigma^0_{2\alpha}(Y)$, there is a Y-computable reduction from B to A_α. In particular, A_α is $\boldsymbol{\Sigma}^0_{2\alpha}$-complete, and if $\alpha < \omega_1^{CK}$, then A_α is $\Sigma^0_{2\alpha}$-complete.

Proof. Let a be a notation such that for all X, $a \in \mathcal{O}^{X\oplus Y}$ and $|a|_{\mathcal{O}}^{X\oplus Y} = 2\alpha$. For all X, by relativization we have

$$H_{2^a}^{X\oplus Y} \equiv_1 \{e : \phi_e^{X\oplus Y} \text{ codes a tree } T \text{ with } |T|_{ls} \leq \alpha\}, \tag{1}$$

where the pair of reductions witnessing the 1-equivalence do not depend on X.[1] Letting d_0 be such that $\phi_{d_0}^{X\oplus Y} = X$, the reverse reduction in (1) provides e_0 such that for all X,

$$X \in A_\alpha \iff \phi_{d_0}^{X\oplus Y} \in A_\alpha \iff e_0 \in H_{2^a}^{X\oplus Y},$$

so $A_\alpha \in \Sigma^0_{2\alpha}(Y)$.

To show that a given $B \in \Sigma^0_{2\alpha}(Y)$ can be Y-computably reduced to A_α, it suffices to consider $B = \{X : n_0 \in H_{2^a}^{X\oplus Y}\}$ where n_0 is chosen to make B $\Sigma^0_{2\alpha}(Y)$-universal. The forward reduction in (1) provides m_0 such that $X \in B \iff \phi_{m_0}^{X\oplus Y} \in A_\alpha$, and the mapping $X \mapsto \phi_{m_0}^{X\oplus Y}$ is Y-computable. □

1.3. **Denjoy totalization.** Classical real analysis includes the study of absolutely continuous functions, functions of bounded variation, and countable generalizations of these notions. We consider four classes of real-valued functions on I. They are VBG (generalized bounded variation), VBG_* (generalized bounded variation in the restricted sense), ACG (generalized absolutely continuous) and ACG_* (generalized absolutely continuous in the restricted sense).

To define these classes it is necessary to generalize the well-known definitions of bounded variation and absolute continuity to take into account also a closed set E to which the function F should be in some sense restricted. The non-asterisk definitions are the literal restrictions. The others take into account also the values of F outside of E. The *oscillation* of a function F on an interval (a,b) is defined as $\omega(F,a,b) = \sup_{x,y\in(a,b)} |F(x) - F(y)|$.

Definition 2. *Let $F \in C(I)$ and let $E \subseteq I$.*

1. *We say F is VB (respectively VB_*) on E if there is an N such that for all non-decreasing sequences $a_0, b_0, \dots a_k, b_k \in E$, we have $\sum_i |F(b_i) - F(a_i)| < N$ (respectively $\sum_i \omega(F, a_i, b_i) < N$).*
2. *We say F is AC (respectively AC_*) on E if for all ε there is a δ such that for all non-decreasing sequences $a_0, b_0, \dots a_k, b_k \in E$, if $\sum_i |b_i - a_i| < \delta$, then $\sum_i |F(b_i) - F(a_i)| < \varepsilon$ (respectively $\sum_i \omega(F, a_i, b_i) < \varepsilon$).*

Observe that if E is an interval, then being VB on E is the same thing as being VB_* on E, and similarly for absolute continuity. We can also understand what it means for a function to satisfy these properties on a closed set E with reference to simplified functions F_E and $F_{E,*}$ defined as follows.

Definition 3. *Let $F \in C(I)$, and $E \subseteq I$ a closed set. Then let F_E and $F_{E,*}$ denote the functions satisfying*

1. *$F_E(x) = F_{E,*}(x) = F(x)$ for $x \in E$, and*
2. *If (c,d) is a connected component of $I \setminus E$,*

[1]Because both sets consist of machine indices for machines with access to $X \oplus Y$, the existence of a single pair of computable reductions follows from the existence of a single pair of uniformly $X \oplus Y$-computable reductions.

(a) *let* F_E *be linear on* $[c,d]$*, and*
(b) *let*

$$F_{E,*}\left(\frac{2c+d}{3}\right) = \sup F([c,d])$$
$$F_{E,*}\left(\frac{c+2d}{3}\right) = \inf F([c,d]),$$

and let $F_{E,*}$ *be linear on* $[c,\frac{2c+d}{3}], [\frac{2c+d}{3},\frac{c+2d}{3}]$ *and* $[\frac{c+2d}{3},d]$.

Note that $\omega(F_{E,*},[c,d]) = \omega(F,[c,d])$ for (c,d) a connected component of $I\setminus E$.
Then the following proposition holds:

Proposition 6. *Let* $F\in C(I)$ *and* $E\subseteq I$ *be closed. Then*

1. F *is* VB *(resp.* AC*) on* E *if and only if* F_E *is* VB *(resp.* AC*) on* I.
2. F *is* VB_* *(resp.* AC_**) on* E *if and only if* $F_{E,*}$ *is* VB *(resp.* AC*) on* I.

Now we can define the main notions.

Definition 4. *A function* $F\in C(I)$ *is* VBG *(respectively* ACG, VBG_*, ACG_**) if there is a countable sequence of closed sets* E_n *such that* $\cup_n E_n = I$ *and* F *is* VB *(respectively* AC, VB_*, AC_**) on each* E_n.

It is immediate that $VBG_*\subseteq VBG$ and $ACG_*\subseteq ACG$. Recall the relationship between absolute continuity and bounded variation: a continuous function of bounded variation is absolutely continuous if and only if it satisfies Lusin's condition (N).

Definition 5. *A function* $F: I\to\mathbb{R}$ *satisfies* (N) *if for every Lebesgue null set* $A\subseteq I$*, its image* $F(A)$ *is also null.*

If $\cup_n E_n = I$, then F satisfies (N) if and only if it satisfies (N) on each E_n. Therefore, $F\in ACG$ if and only if $F\in VBG$ and F satisfies (N). We also have (see [Sak64, Thm VII.8.8, pg. 233]) that $ACG_* = VBG_*\cap ACG$. It follows that $F\in ACG_*$ if and only if $F\in VBG_*$ and F satisfies (N).

Based on the definitions above, it would seem that these sets are Σ^1_2. However, the following equivalent characterization shows each of these classes is in fact Π^1_1.

Theorem 7 (see [Sak64, Thm VII.9.1, pg 233])**.** *For a function* F *to be* VBG *(respectively* VBG_*, ACG, ACG_**) it is necessary and sufficient that for every closed* $E\subseteq I$*, there is an interval* $[a,b]\subseteq I$ *such that* $(a,b)\cap E\neq\emptyset$ *and* F *is* VB *(respectively* VB_*, AC, AC_**) on* $[a,b]\cap E$.

Corollary 8. *The sets* $VBG, VBG_*, ACG, ACG_*\subseteq C(I)$ *are all* Π^1_1.

The previous theorem also suggests a derivation process.

Definition 6. *Let* X *stand for* VB, VB_*, AC *or* AC_**. Given* $F\in C(I)$*, define* $P^0_{F,X} = I$*. Define*

$$P^{\alpha+1}_{F,X} = P^{\alpha}_{F,X}\setminus\cup\{(a,b) : F \text{ is } X \text{ on } [a,b]\cap P^{\alpha}_{F,X}\}$$

and for a limit ordinal λ*, define* $P^{\lambda}_{F,X} = \cap_{\alpha<\lambda} P^{\alpha}_{F,X}$*. Define a rank* $|\cdot|_X$ *by letting* $|F|_X$ *be the least* α *such that* $P^{\alpha}_{F,X}=\emptyset$*, if such* α *exists.*

If $F \in VBG$, then the only way for $P^{\alpha+1}_{F,VB} = P^{\alpha}_{F,VB}$ is if $P = \emptyset$; and furthermore, the countable sequence of sets $[a,b] \cap P^{\alpha}_{F,VB}$ for which (a,b) were removed over the course of the derivations would serve as the sequence E_n required in the definition of VBG. Reasoning similarly about all four hierarchies, it is immediate that a given $F \in C(I)$ belongs to one of the classes VBG, VBG_*, ACG, ACG_* if and only if the associated derivation process eventually produces the empty set.

Definition 7. *Let VBG_α (respectively $VBG_{*\alpha}, ACG_\alpha, ACG_{*\alpha}$) denote the sets $\{F \in C(I) : |F|_{VB} \leq \alpha\}$ (respectively $|F|_{VB_*}, |F|_{AC}, |F|_{AC_*}$).*

Recall from the introduction that ACG_* is exactly the set of functions $F \in C(I)$ which can be recovered from F' by Denjoy totalization. We will not give the definition of the Denjoy totalization process; the interested reader may consult [Sak64, Section VIII.5]. What matters for us is that Denjoy totalization is a transfinite procedure which terminates at some countable ordinal stage, and $ACG_{*\alpha}$ consists of precisely the functions $F \in ACG_*$ which are recovered from F' in at most α steps of Denjoy totalization. Therefore, the sets $ACG_{*\alpha}$ have a meaningful interpretation in terms of Denjoy totalization.

Note that there are actually two transfinite procedures which are sometimes called "Denjoy totalization": the narrow Denjoy integral, which coincides with the integrals of Perron, Kurzweil and Henstock, and the wide Denjoy integral, sometimes known as the Denjoy-Khintchine integral. In this paper, "Denjoy totalization" always refers to the narrow Denjoy integral. The narrow Denjoy integral has the same relationship to the class ACG_* as the wide Denjoy integral has to the class ACG.

2. Cantor-Bendixson rank

In this section we analyze the initial segments of the Cantor-Bendixson hierarchy. The theorems of this section are not used in later sections. Let $S \subseteq 2^{<\omega}$ be a tree with no dead ends. Let $[S]$ denote the set of paths in S. The Cantor-Bendixson derivative $D(S)$ is defined as the tree without dead ends such that $[D(S)]$ consists of exactly the paths not isolated in $[S]$. Define $D^0(S) = S$, $D^{\alpha+1}(S) = D(D^\alpha(S))$, and $D^\lambda(S) = \cap_{\alpha<\lambda} D^\alpha(S)$ for λ a limit.

Definition 8. *The Cantor-Bendixson rank of a tree $S \subseteq 2^\omega$, denoted $|S|_{CB}$, is the least α such that $D^\alpha(S) = \emptyset$, if such exists. Otherwise we say $|S|_{CB} = \infty$.*

Other authors define $|T|_{CB}$ to be the least α such that $D^\alpha(S) = D^{\alpha+1}(S)$, so that a set with a perfect subset also has a Cantor-Bendixson rank. Others define their rank to be always one less than ours, so that every ordinal is used.

Proposition 9. *Let $Y \in 2^\omega$ and $\alpha < \omega_1^Y$. Then $\{S : |S|_{CB} \leq \alpha\}$ is $\Sigma^0_{2\alpha}(Y)$ in $\{S : S$ is a tree with no dead ends$\}$.*

Proof. The proof is by effective transfinite recursion. Because checking whether a no-dead-ends tree is empty can be accomplished by checking the root, $\{S : |S|_{CB} = 0\}$ is Σ^0_0.

A tree S has $D^{\alpha+1}(S) = \emptyset$ if and only if $D^\alpha(S)$ has only finitely many branches. If $D^\alpha(T)$ has at least k branches, then by going up to a height n at which the branches have separated, we may find at least k-many σ of length n such that $D^\alpha(T_\sigma) \neq \emptyset$. And if there are k incomparable σ such that $D^\alpha(T_\sigma) \neq \emptyset$, then $D^\alpha(T)$ has at least k branches. Thus

$$D^{\alpha+1}(S) = \emptyset \iff \exists k \forall n (\text{there are at most } k \text{ many } \sigma \text{ of length } n \text{ for which } D^\alpha(S_\sigma) \neq \emptyset).$$

Assuming $\{S : D^\alpha(S) = \emptyset\}$ is $\Sigma^0_{2\alpha}(Y)$ uniformly in α, this shows that $D^{\alpha+1}(S) = \emptyset$ is $\Sigma^0_{2\alpha+2}(Y)$ uniformly in α.

If λ is a limit, a tree has $D^\lambda(S) = \emptyset$ if and only if there is an $\alpha < \lambda$ such that $D^\alpha(S) = \emptyset$, by compactness. Assuming $D^\alpha(S) = \emptyset$ is $\Sigma^0_{2\alpha}(Y)$ uniformly in $\alpha < \lambda$, and supposing a sequence $\alpha_n \to \lambda$ is Y-effectively given, we have $D^\lambda(S) = \emptyset$ if and only if $\exists \alpha < \lambda[D^\alpha(S) = \emptyset]$, a $\Sigma^0_\lambda(Y)$ statement. Note $\Sigma^0_\lambda(Y) = \Sigma^0_{2\lambda}(Y)$ for λ a limit. □

Proposition 10. *There is a computable reduction $T \mapsto S_T$ from trees $T \subseteq \omega^{<\omega}$ to no-dead-end trees $S \subseteq 2^{<\omega}$, satisfying*

(1) *If T is not well-founded, $[S_T]$ contains a perfect set.*
(2) *If T is well-founded with $|T|_{ls} = \alpha$, then $[S_T]$ is countable, and $|S_T|_{CB} = \alpha$.*

Proof. The idea is that each node of a tree $T \subseteq \omega^{<\omega}$ should correspond to a branching of paths in $S_T \subseteq 2^{<\omega}$, with the topological clustering of the paths provided by the hierarchical structure of T.

Define S_T by

$$S_T = \{0^{2n_0+i_0}10^{2n_1+i_1}1\cdots 0^{2n_k+i_k}10^m : (n_0, \ldots, n_k) \in T, i_0, \ldots i_k \in \{0, 1\}\}.^2$$

Note that $0^m \in S_T$ if and only if the empty node is in T. If $T \notin WF$, then if $P \in [T]$ the following subset of $[S_T]$ is perfect:

$$\{0^{2P(0)+X(0)}10^{2P(1)+X(1)}1\cdots : X \in 2^\omega\}.$$

Supposing now that T is well-founded, we claim that $|T|_{ls} = |S_T|_{CB}$. The proof is by induction on the usual rank of T. If $T = \emptyset$ then also $S_T = \emptyset$, so $|S_T|_{CB} = |T|_{ls} = 0$.

Suppose $|T|_{ls} = \alpha+1$. (This is the only case because the limsup rank is always a successor.) Then there is an N such that for $n \geq N$, $|T_n|_{ls} \leq \alpha$. Observe that if $C \subseteq 2^{<\omega}$ is any finite prefix-free collection of strings such that $\cup_{\sigma\in C}[\sigma]$ covers $[S]$, then

$$D^\alpha(S) = \bigcup_{\sigma\in C} \sigma^\frown D^\alpha(S_\sigma).$$

Letting $C = \{0^{2N}\} \cup \{0^{2n+i}1 : n < N, i \in \{0, 1\}\}$, we have

$$D^{\alpha+1}(S_T) = \bigcup_{\sigma\in C} \sigma^\frown D^{\alpha+1}((S_T)_\sigma).$$

By induction, for $n < N$ and $i \in \{0, 1\}$, $|0^{2n+i}1^\frown S_{T_n}|_{CB} \leq \alpha + 1$, so

$$D^{\alpha+1}(S_T) = 0^{2N\frown} D^{\alpha+1}((S_T)_{0^{2N}}).$$

Also, for $n \geq N$, we have $|0^{2n+i}1^\frown S_{T_n}|_{CB} \leq \alpha$, so for each k, $D^\alpha((S_T)_{0^{2N+k}1}) = \emptyset$, so $[D^\alpha((S_T)_{0^{2N}})] \subseteq \{0^\omega\}$. Therefore $D^{\alpha+1}((S_T)_{0^{2N}}) = \emptyset$, so $|S_T|_{CB} \leq \alpha + 1$.

Now we need $|S_T|_{CB} \geq \alpha+1$. If $|T_n|_{ls} = \alpha+1$ for some n, then by induction $|0^n1^\frown S_{T_n}|_{CB} = \alpha + 1$, which suffices. If $\limsup_n |T_n|_{ls} = \alpha$, then for every $\beta < \alpha$, there are infinitely many n such that $|T_n| > \beta$, so there are infinitely many n for which $|0^n1^\frown S_{T_n}|_{CB} > \beta$. Therefore, for all $\beta < \alpha$, 0^ω is not an isolated path of $D^\beta(S_T)$, so $0^\omega \in [D^\alpha(S_T)]$, and $|S_T|_{CB} > \alpha$. □

[2] The more familiar option, $S'_T = \{0^{n_0}10^{n_1}1\cdots 0^{n_k}10^m : (n_0, \ldots, n_k) \in T\}$, satisfies part (2) but not part (1) of the theorem; consider the case when $[T]$ consists of a single path.

Theorem 11. *For any nonzero $\alpha < \omega_1$, let*

$$A_\alpha = \{S \subseteq 2^\omega : S \text{ is a no dead end tree and } |S|_{CB} \leq \alpha\}.$$

Then all the conclusions of Theorem 5 hold. In particular, if $\alpha < \omega_1^{CK}$, A_α is $\Sigma^0_{2\alpha}$-complete.

A special case is the following.

Corollary 12. [Lem87] *For each constructive $\alpha > 1$, the sets*

$$\{e : \phi_e \text{ codes a tree } S \text{ which has no dead ends and } |S|_{CB} \leq \alpha\}$$

are $\Sigma_{2\alpha}$-complete.

This result differs only cosmetically from the result as it was stated in in [Lem87] because he considered trees which might have dead ends and because he used a definition of $\Sigma^0_{2\alpha}$-complete which is off by one from our definition for $\alpha \geq \omega$.

3. Completeness and hardness on the Denjoy hierarchies

In this section we present a construction which provides a computable reduction from $(WF, \neg WF)$ to $(ACG_*, \neg VBG)$. Because $ACG_* \subseteq ACG \cap VBG_*$ and $ACG \cup VBG_* \subseteq VBG$, this shows that all four classes are Π^1_1-complete. Additionally, this reduction serves as a simultaneous uniform reduction from $(A_\alpha, A_{\alpha+1})$ to $(ACG_{*\alpha}, \neg VBG_\alpha)$, where $A_\alpha = \{T \in WF : |T|_{ls} \leq \alpha\}$. We have $ACG_{*\alpha} \subseteq ACG_\alpha \cap VBG_{*\alpha}$ and $ACG_\alpha \cup VBG_{*\alpha} \subseteq VBG_\alpha$, so all these sets are at least as complex as A_α, namely at least $\Sigma^0_{2\alpha}(Y)$-hard.

The idea is that each node of the given tree should contribute a finite length to the variation of the constructed function. In most cases the total variation will be infinite as a result, but the way in which that infinite length is distributed will determine the rank of the function.

The following notation will be useful: if $J, K \subseteq I$ are two intervals, then $J\langle K\rangle$ denotes the interval that has the same relation to K as J has to I; that is, if $J = [a, b]$ and $K = [c, d]$ then $J\langle K\rangle = [c + a(d - c), c + b(d - c)]$.

To define this reduction, it will be useful to have a way to tell F_T to increase its variation in a given interval J in response to seeing more of T. Given an interval $J \subseteq [0, 1]$, we define the wiggle function $W(J)$ as follows. The goal is to have a function whose variation is at least 1 (regardless of how small J is), but which only takes values in $[0, |J|]$, where $|J|$ is the length of the interval J. Therefore, for small J, the function should oscillate intensely. We also leave some space at the top and bottom of each oscillation to give room for adding some more oscillations; this sets us up for a typical method of producing a function of high Denjoy rank.

Definition 9. *Given an interval $J \subseteq I$, let M be the least integer large enough that $2M|J| > 1$, and define*

1. $W(J)(x) = 0$ *if* $x \notin J$.
2. $W(J)(x) = 0$ *if* $x \in [\frac{4k}{4M+1}, \frac{4k+1}{4M+1}]\langle J\rangle$ *for* $k \leq M$.
3. $W(J)(x) = |J|$ *if* $x \in [\frac{4k+2}{4M+1}, \frac{4k+3}{4M+1}]\langle J\rangle$ *for* $k < M$.
4. *If* $x \in [\frac{4k+i}{4M+1}, \frac{4k+i+1}{4M+1}]\langle J\rangle$ *for* $k < M$ *and* $i = 1, 3$, *determine the value of* $W(J)(x)$ *by linear interpolation from the values already defined.*

Define

$$\begin{aligned}\mathrm{Flat}(J) &= \{K \subseteq J : K \textit{ is a maximal interval on which } W(J) \textit{ is constant}\} \\ &= \left\{ \left[\frac{4k+i}{4M+1}, \frac{4k+i+1}{4M+1}\right] \langle J \rangle : k \leq M, i = 0, 2 \right\}\end{aligned}$$

Fix an infinite sequence of disjoint closed interval subsets of I, decreasing in size as they approach 0, with gaps between them. For specificity, say $J_n = [\frac{1}{3n+2}, \frac{1}{3n+1}]$. For any interval L, we define the function $\tilde{F}(T, L)$ by recursion on the usual rank of T as follows.

Definition 10. *Given an interval $L \subseteq I$, let $\tilde{F}(\emptyset, L)$ be the constant function 0, and for nonempty T, let*

$$\tilde{F}(T, L) = W(L) + \sum_{K \in \mathrm{Flat}(L), n \in \omega} \tilde{F}(T_n, J_n\langle K \rangle).$$

Observe that the $\tilde{F}(T_n, J)$ are being copied onto the plateaus of the wiggle functions.

It is possible to extend this recursive definition of $\tilde{F}$ so that any tree T (even ill-founded) can be used. Each $\sigma \in T$ contributes some wiggle to $\tilde{F}(T, L)$, in locations which can be described as follows. Let $\mathcal{C}^L_\emptyset = \{L\}$, and given C^L_σ for $|\sigma| \geq 1$, define

$$C^L_{\sigma^\frown n} = \bigcup_{H \in \mathcal{C}^L_\sigma} \{J_n\langle K \rangle : K \in \mathrm{Flat}(H)\}.$$

Definition 11. *Given an interval $L \subseteq I$, define*

$$F(T, L) = \sum_{\substack{\sigma \in T \\ H \in C^L_\sigma}} W(H).$$

Proposition 13. *For all trees $T \subseteq \omega^{<\omega}$, $F(T, L)$ is well-defined and uniformly T-computable. For $T \in WF$, $F(T, L) = \tilde{F}(T, L)$.*

Proof. We may decompose $F(T, L) = \bigcup_{\ell < \omega} F_\ell(T, L)$, where

$$F_\ell(T, L) = \sum_{\substack{\sigma \in T : |\sigma| = \ell \\ H \in C^L_\sigma}} W(H),$$

Observe that since each J_n satisfies $|J_n| < 1/2$, each $H \in C^L_\sigma$ satisfies $|H| < 2^{-|\sigma|}$. Furthermore, the intervals of $\bigcup_{\sigma : |\sigma| = \ell} C^L_\sigma$ are disjoint. Therefore, since $W(H)$ has its range in $[0, |H|]$, each $F_\ell(T, L)$ has its range in $[0, 2^{-\ell}]$. Also, the $F_\ell(T, L)$ are uniformly computable. Therefore, $F(T, L)$ is the effective uniform limit of computable functions, and is therefore computable.

Now suppose $T \in WF$. If T is just a root, it is clear that $F(T, L) = \tilde{F}(T, L)$. Assuming that each $F(T_n, K) = \sum_{\sigma \in T_n, H \in C^K_\sigma} W(H)$, the agreement of the two definitions follows in general because for each n,

$$C^L_{n^\frown \sigma} = \bigcup_{K \in \mathrm{Flat}(L)} C^{J_n\langle K \rangle}_\sigma.$$

□

Proposition 14. *If T is not well-founded, then $F(T, I) \notin VBG$.*

Proof. Let Z be a path in T, and consider

$$P_Z = \bigcap_{\ell\in\omega} \left(\bigcup \mathcal{C}^I_{Z\restriction\ell}\right)$$

This is an intersection of a decreasing sequence of closed sets in a compact space, so P_Z is closed and non-empty. We claim that there is no open interval U such that $P_Z \cap U$ is nonempty and $F(T,I)$ has bounded variation on $P_Z \cap U$. By Theorem 7, this suffices.

Fix an open interval U and a number N. Since $P_Z \cap U \neq \emptyset$, there is an M and $H_0 \in \mathcal{C}_{Z\restriction M}$ such that $P_Z \cap H_0 \subseteq U$. We will show that the variation of $F(T,I)$ on $P_Z \cap H_0$ is at least 2^N. Let $\mathcal{C} = \{H \in \mathcal{C}_{Z\restriction(M+N)} : H \subseteq H_0\}$. Since each interval of $\mathcal{C}^I_{Z\restriction\ell}$ is broken into multiple intervals in $\mathcal{C}^I_{Z\restriction(\ell+1)}$, there are at least 2^N intervals in $\mathcal{C}$.

Let the functions F_ℓ be as in the previous proposition. For each $H \in \mathcal{C}$, $\sum_{\ell<M+N} F_\ell(T,I)$ is constant on H, the function $W(H)$ has variation at least 1, and $F_{M+N}(T,I)(x) = W(H)(x)$ for all $x \in H$. Therefore, the function G defined by $G = \sum_{\ell\leq M+N} F_\ell(T,I)$ has variation at least 1 on H. In fact, G has variation at least 1 on any subset of H which contains at least one point of K for each $K \in \mathrm{Flat}(H)$. Therefore, it suffices to show that each such K contains a point $x \in P_Z$ such that $G(x) = F(T,I)(x)$.

We claim that $x = \min(P_Z \cap K)$ is such a point. For each $\ell > M+N$, there is a unique interval $J \in \cup_{\sigma\in T:|\sigma|=\ell}\mathcal{C}^I_\sigma$ such that $x \in J$. All those intervals are disjoint, so $F_\ell(T,I)(x) = W(J)(x)$. We claim that $W(J)(x) = 0$. Since $x \in P_Z$, $J \in \mathcal{C}^I_{Z\restriction\ell}$. Because x is the minimum of $P_Z \cap K$, it is also the minimum of $P_Z \cap J$. But P_Z contains points from every $L \in \mathrm{Flat}(J)$. Therefore, x is an element of the first such interval $L \subseteq J$. Since $W(J) \equiv 0$ on its first interval, $W(J)(x) = 0$, as required. Since this is true for arbitrary ℓ, we have $G(x) = F(T,I)(x)$ for all such x. Therefore, the variation of $F(T,I)$ on H_0 is at least 2^N, so $F(T,I) \notin VBG$. □

Proposition 15. *If T is well-founded and $|T|_{ls} \geq 1$, then $F(T,I) \in ACG_*$, and $|F(T,I)|_X = |T|_{ls}$, where X is any of VB, VB_*, AC, AC_*.*

Proof. We proceed by induction on the usual rank of the tree, starting with the root-only tree $T = \{\emptyset\}$. The statement we prove inductively is slightly stronger: for all intervals L, if $|T|_{ls} = \alpha+1$ then $|F(T,L)|_X = \alpha+1$, and furthermore there is a point $x \in P^\alpha_{F(T,L),X}$ such that $F(T,L)(x) = 0$. From here forward, we drop the subscripts on $P^\alpha_{F(T,L),X}$, whenever possible, writing instead P^α. When subscripts are included, it is because we reference the derivation applied to a different function, typically $F(T_n,K)$ for some n and some K.

In the base case, the tree $T = \{\emptyset\}$ has $|T|_{ls} = 1$, and $F(T,L) = W(L)$, so it also has rank 1. Let T be given, with $|T|_{ls} = \alpha+1$. By induction, assume that for each n and L', $|T_n|_{ls} = |F(T_n,L')|_X$. We know that

$$P^\alpha \subseteq \bigcup_{K\in\mathrm{Flat}(L)} \{\min K\} \cup \bigcup_n J_n\langle K\rangle$$

because $F(T,L)$ vanishes outside this set. Because $|T_n|_{ls} \leq \alpha+1$ for all n, we have $P^{\alpha+1} \cap J_n\langle K\rangle = \emptyset$ for all n. Similarly, because $|T_n|_{ls} \leq \alpha$ for sufficiently large n, we also have $P^\alpha \cap J_n\langle K\rangle = \emptyset$ for sufficiently large n. Therefore, if $\min K \in P^\alpha$, it is isolated, so $\min K \notin P^{\alpha+1}$. This shows that $|F(T,L)|_X \leq \alpha+1$.

On the other hand, suppose that $|T|_{ls} = \alpha+1$ because $|T_n|_{ls} = \alpha+1$ for some n. Then, letting $K \in \mathrm{Flat}(L)$ be leftmost, and letting $x \in P^\alpha_{F(T_n,J_n\langle K\rangle),X}$ with $F(T_n, J_n\langle K\rangle)(x) =$

0, observe that $x \in P^\alpha$ as well, and $F(T,L)(x) = W(L)(x) + F(T_n,K)(x) = 0$, where $W(L)(x) = 0$ because K was chosen leftmost.

Finally, suppose that $|T|_{ls} = \alpha + 1$ because for every $\beta < \alpha$, there are infinitely many n such that $|T_n|_{ls} > \beta$. Again let $K \in \mathrm{Flat}(L)$ be leftmost. For each such β, we have $\min K \in P^\beta$, because for infinitely many n, $P^\beta \cap J_n\langle K\rangle \neq \emptyset$. If α is a limit, it is now immediate that $\min K \in P^\alpha$. Suppose that $\alpha = \beta + 1$. We claim that $F(T,L)$ is not VB on P^β in any neighborhood of $\min K$, which also shows that $\min K \in P^\alpha$. In either case, observe that $F(T,L)(\min K) = 0$.

To prove the claim, given N and a neighborhood $U \ni \min K$, let $n_1 < n_2 \cdots < n_N$ be chosen so that $J_{n_i}\langle K\rangle \subseteq U$ and $|T_{n_i}|_{ls} = \alpha$. Then $\sum_{i=1}^N W(J_{n_i}\langle K\rangle)$ has variation at least N. If $P \subseteq K$ is any set containing at least one point from each $H \in \mathrm{Flat}(J_{n_i}\langle K\rangle)$ for each $i \leq N$, then $\sum_{i=1}^N W(J_{n_i}\langle K\rangle)$ has variation at least N on$P$. By the choice of n_i, P^β is such a set, and for each $H \in \mathrm{Flat}(J_{n_i}\langle K\rangle)$, there is $x \in P^\beta \cap H$ such that $F(T_{n_i}, J_{n_i}\langle K\rangle)(x) = 0$. Therefore, these points suffice to witness that $F(T,L)$ has variation at least N on $P^\beta \cap U$, because $F(T,L)(x) = \sum_{i=1}^N W(J_{n_i}\langle K\rangle)(x)$ for all such x. Therefore, $|F(T,L)|_X = \alpha + 1$ and $\min K \in P^\alpha$ satisfies $F(T,L)(\min K) = 0$. □

Corollary 16. *Let $Y \in 2^\omega$ and $\alpha < \omega_1^Y$. For any $\Sigma^0_{2\alpha}(Y)$ set B, there is a Y-computable reduction from $(B, \neg B)$ to $(ACG_{*\alpha}, \neg VBG_\alpha)$.*

Theorem 17. *The sets VBG, VBG_*, ACG, ACG_* are all Π^1_1-complete.*

This answers a question of Walsh [Wal17], who asked whether ACG_* was Π^1_1-complete. Walsh also showed that the graph of Denjoy integration is a Π^1_1, non-Borel subset of $M(I) \times C(I)$. He asked whether that set is Π^1_1-complete, a question which we answer in the affirmative. From here on, let $F_T = F(T, I)$.

Lemma 18. *For all trees $T \subseteq \omega^{<\omega}$, F_T is a.e differentiable, and the map $T \mapsto (F'_T, F_T) \in M(I) \times C(I)$ is computable.*

Proof. For each ℓ, let

$$G_\ell = \sum_{\substack{\sigma \in T : |\sigma| < \ell \\ \text{and } \max(\sigma) < \ell \\ H \in \mathcal{C}^I_\sigma}} W(H).$$

Then $\lim_{\ell\to\infty} G_\ell = F_T$. Observe that G'_ℓ is a.e. equivalent to an ideal point of $M(I)$. We claim that in $M(I)$, $\lim_{\ell\to\infty} G'_\ell = F'_T$. For this it suffices to observe that $G_\ell = F_T$ on any interval where $G'_\ell \neq 0$, and also on any interval that is disjoint from all intervals of $\mathcal{C}^I_\sigma$ for $\sigma \in T$ with $|\sigma| \geq \ell$ or $\max(\sigma) \geq \ell$. Regardless of whether T is well-founded, the measure of these intervals of agreement approaches 1 in the limit, and the convergence is effective. □

Theorem 19. *The set $\{(f,F) \in M(I) \times C(I) : F \in ACG_*$ and $F' = f$ a.e.$\}$ is Π^1_1-complete.*

Proof. The map $T \mapsto (F'_T, F_T)$ provides a computable reduction from WF to $\{(f,F) \in M(I) \times C(I) : F \in ACG_*$ and $F' = f$ a.e.$\}$. □

The next and final result of this section concerns $\{f \in M(I) : f$ is Denjoy integrable$\}$. Walsh showed that this set is $\boldsymbol{\Sigma}^1_2$ and not $\boldsymbol{\Sigma}^1_1$, and asked for better bounds, which we give in Theorem 21. We need a lemma.

Lemma 20. *If $T \notin WF$, then F'_T is not Denjoy integrable.*

Proof. Let $P = \cap_{\ell\in\omega}\overline{\cup D_\ell}$, where

$$D_\ell = \{H : H \in \mathcal{C}^I_\sigma, \sigma \in T, |\sigma| = \ell \text{ and } T_\sigma \notin WF\}$$

Then F_T is not VB on $P \cap U$ for any open interval U such that $P \cap U \neq \emptyset$, by a modification of the argument in Proposition 14. Let $H_0 \subseteq U$ with $H_0 \in D_M$ for some M, and let $H \subseteq H_0$ with $H \in D_{M+N}$. There are at least 2^N such H, all disjoint. Then as before, the variation of F_T on each such H is at least 1. Furthermore, if $\text{Flat}(H) = \{K_0, K_1, \ldots, K_r\}$, listed from left to right, then this variation is witnessed by the intervals $(\max(K_i \cap P), \min(K_{i+1} \cap P))$ for each $i < r$, and these intervals are disjoint from P.

We claim that the restriction of F_T to any closed interval J disjoint from P is ACG_*. Such an interval is disjoint from $\cup D_\ell$ for some ℓ, and therefore $F_T \restriction J = F_S \restriction J$, where S is the well-founded tree obtained from T by removing all σ such that $T_{\sigma\restriction\ell} \notin WF$. Because F_S is ACG_*, $F_T \restriction J$ is ACG_*.

Now suppose, for the sake of contradiction, that there is $G \in ACG_*$ with $G' = F_T'$. Then G would be VB on some portion of P. But for each connected component (c, d) of $I \setminus P$, we have $G(d) - G(c) = F_T(d) - F_T(c)$ (because, using the continuity of F_T and G, both are equal to the limit as J expands to (c, d) of the Denjoy integral of $F_T' \restriction J$), and F_T is not VB on any portion of P, via a sequence that uses such intervals (c, d) as witnesses. Such a sequence also witnesses that G has variation at least 2^N on $P \cap U$. □

Theorem 21. *The set $\{f \in M(I) : f$ is Denjoy integrable $\}$ is Π^1_1-complete.*

Proof. Dougherty and Kechris [DK91, pg. 162] present a proof, originally due to Ajtai, that for sequences $\overline{f} \in C(I)^\omega$, if $\overline{f}$ converges to f and f is the derivative of an everywhere differentiable function F, then $F \in \Delta^1_1(\overline{f})$. An inspection of the proof shows that the same argument works if the limiting function f is given as $f \in M(I)$ (i.e. just its a.e. equivalence class is given) and all that is needed is for f to be Denjoy integrable (so that $F \in ACG_*$ with $F' = f$, rather than F being everywhere differentiable). Therefore, for $f \in M(I)$,

$$f \text{ is Denjoy integrable} \iff \exists F \in \Delta^1_1(f)[F' = f \text{ a.e.}]$$

which is Π^1_1.

Now consider the completeness direction. By Lemma 18, the map $T \mapsto F_T'$ is computable, and $T \in WF$ if and only if F_T' is Denjoy integrable by Lemma 20. □

4. Descriptive results on the Denjoy hierarchies

4.1. **Descriptive complexity of VBG and VBG_* hierarchies.** Our goal now is to give the precise descriptive complexity of the sets $VBG_{*\alpha}, VBG_\alpha$ and $ACG_{*\alpha}$. For ACG_α, we will only give an upper bound on the descriptive complexity.

Proposition 22. *If $\alpha < \omega_1^Y$, then $VBG_{*\alpha}$ and VBG_α are $\Sigma_{2\alpha}(Y)$.*

Proof. By effective transfinite recursion. Let X be one of VB, VB_*, and let $p, q \in \mathbb{Q}$. When $\alpha = 1$, we have $P^1_{F,X} \cap (p, q) = \emptyset$ if and only if F has bounded variation on the interval $[p, q]$. This property is Σ^0_2 uniformly in F, p and q.

Supposing that $\{(F, p, q) : [p, q] \cap P^\alpha_{F,X} = \emptyset\}$ is $\Sigma^0_{2\alpha}(Y)$ uniformly in α, let us show $\{(F, p, q) : [p, q] \cap P^{\alpha+1}_{F,X} = \emptyset\}$ is $\Sigma_{2\alpha+2}(Y)$ uniformly in α. We have by Proposition 6,

$$P^{\alpha+1}_{F,VB} \cap [p, q] = \emptyset \iff \exists [p', q'] \supsetneq [p, q] \text{ such that } F_{P^\alpha} \text{ is } VB \text{ on } [p', q']$$

$$P^{\alpha+1}_{F,VB_*} \cap [p, q] = \emptyset \iff \exists [p', q'] \supsetneq [p, q] \text{ such that } F_{E,*} \text{ is } VB \text{ on } [p', q'], \text{ where } E = P^\alpha.$$

So it is sufficient to show that there is a uniform procedure for computing, given ε, approximations to F_{P^α} and $F_{P^\alpha,*}$ that are correct to within ε, using an oracle which can answer $\Sigma^0_{2\alpha}(F,Y)$ questions, for example questions of the form "$P^\alpha_{F,X} \cap [a,b] = \emptyset$?"

Using F and the compactness of I, compute a number N large enough that

$$\omega(F,[k/2^N,(k+1)/2^N]) < \varepsilon$$

for all $k < 2^N$. For each $\ell < k \leq 2^N$, ask whether $P^\alpha_{F,X} \cap [\ell/2^N, k/2^N] = \emptyset$. This identifies (up to an error of $1/2^N$ on each side) the connected components of $I \setminus P^\alpha_{F,X}$. Let E be the corresponding approximation to $P^\alpha_{F,X}$; that is,

$$E = I \setminus \cup\{(\ell/2^N, k/2^N) : P^\alpha_{F,X} \cap [\ell/2^N, k/2^N] = \emptyset\}.$$

Let F^ε be an approximation of F correct to within ε. Then return F^ε_E or $F^\varepsilon_{E,*}$ as appropriate. The returned function will be correct to within ε on E; on components of $I \setminus E$ that were too small to find, $F^\varepsilon_{E,*}$ could be off by at most 4ε; on components of $I \setminus E$ whose endpoints were only approximated, $F^\varepsilon_{E,*}$ could be off by at most 5ε. The errors for F^ε_E are less.

Therefore, by induction, $\{(F,p,q) : P^{\alpha+1}_{F,X} \cap [p,q] = \emptyset\}$ is $\Sigma^0_{2\alpha+2}(Y)$, uniformly in α.

Finally, if a limit ordinal λ is given as a Y-effective sequence α_n with $\lim_{n\to\infty} \alpha_n = \lambda$, then $[p,q] \cap P^\lambda_{F,X} = \emptyset \iff \exists n [p,q] \cap P^{\alpha_n} = \emptyset$. Since the statements $[p,q] \cap P^{\alpha_n}_{F,X}$ are uniformly $\Sigma^0_{2\alpha_n}(Y)$, we have $\{(F,p,q) : [p,q] \cap P^\lambda_{F,X} = \emptyset\}$ is $\Sigma^0_\lambda(Y) = \Sigma^0_{2\lambda}(Y)$, uniformly in λ. □

Observe that in the proof of Proposition 22, the only place it is used that X is VB or VB_*, rather than AC or AC_*, is in counting the quantifiers of bounded variation in the successor step, to conclude that two jumps on an oracle for "$P^\alpha_{F,X} \cap [a,b] = \emptyset$?" questions would suffice to answer $P^{\alpha+1}_{F,X} \cap [a,b] = \emptyset$ questions. If X were AC or AC_*, this argument gives a bound of 4 jumps. In the case of AC_*, in the next section we will improve the bound to 2 jumps.

Corollary 23. *If $\alpha < \omega_1^Y$, $ACG_{*\alpha}$ and ACG_α are $\Sigma^0_{4\alpha}(Y)$.*

4.2. **Descriptive complexity of the ACG_* hierarchy.** Next we consider the descriptive complexity of ACG_* hierarchy. Already at the first level there is a difference, because $P^1_{F,AC_*} = \emptyset$ if and only if F is absolutely continuous, a Π^0_3 property.

Theorem 24. *The set $\{F \in C(I) : F$ is absolutely continuous$\}$ is Π^0_3-complete.*

Proof. Our strategy is to define a computable reduction from 2^ω to $C(I,I)$ whose output approximates a version of the Cantor function (Devil's Staircase function) which will converge to a Cantor-like function only if the input is an element of a given Π^0_3-complete subset of 2^ω. Let $A \subseteq 2^\omega$ be such a set and let $g(n)$ be a computable function such that for all X,

$$X \in A \iff \forall n [W^X_{g(n)} \text{ is finite}]$$

We now define a function $F : [0,1] \to [0,1]$, uniformly in X, such that F is absolutely continuous if and only if $\forall n [W^X_{g(n)}$ is finite$]$ holds.

Effective in X, we define a computable sequence of functions F_s which converge effectively and uniformly to the desired computable function F. Let $F_0(x) = x$. Each F_s will be piecewise linear, containing some pieces of slope zero separated by pieces of positive slope. Wherever F_s is piecewise constant, it is equal to the limiting function F.

For each n let $I_n = [\frac{1}{n+2}, \frac{1}{n+1}]$. This is the interval in which $W^X_{g(n)}$'s finiteness or lack thereof will be expressed. At stage $s+1$, let $F_{s+1} \upharpoonright [\frac{1}{n+2}, \frac{1}{n+1}] = F_s \upharpoonright [\frac{1}{n+2}, \frac{1}{n+1}]$ for all $n \geq s$ and for all $n < s$ such that no new element of $W^X_{g(n)}$ has been enumerated at stage s. For those n for which a new element is enumerated into $W^X_{g(n)}$, define $F_{s+1} \restriction I_n$ as follows. For each maximal interval $I \subseteq I_n$ on which F_s is constant, let $F_{s+1} \equiv F_s$ on I. For each maximal interval I on which F_s is linear with positive slope, define F_{s+1} on I to satisfy:

(1) $F_{s+1} = F_s$ at the endpoints of I
(2) F_{s+1} is piecewise linear, continuous, and increasing
(3) F_{s+1} has slope zero on $\frac{1}{3}$ of the measure of I, and positive slope everywhere else on I.
(4) F_s and F_{s+1} differ by no more than 2^{-s} at any point.

This can be accomplished by letting $F_{s+1} \restriction I$ resemble a sufficiently fine staircase. The effect is that $\frac{1}{3}$ of the measure of I is given to points at which $F'(x) = 0$. Thus if F'_s was nonzero on a measure r subset of I_n, then F'_{s+1} is nonzero on a measure $\frac{2}{3}r$ subset of I_n.

This completes the construction. One may check that F is continuous and of bounded variation.

Now suppose that it holds that $\forall n[W^X_{g(n)}$ is finite$]$. Then for each n, there will come a stage s for which $F_s \restriction I_n = F \restriction I_n$, and so the final F is piecewise linear on I_n for all n. Then F satisfies the Lusin (N) property because each I_n satisfies it, and there are only countably many I_n. Thus F is absolutely continuous.

On the other had, suppose that $W^X_{g(n)}$ is infinite for some fixed n. Then letting $Z = \cup_s\{x \in I_n : F'_s(x) = 0\}$, we have $\mu(Z) = \mu(I_n)$, but $F(Z)$ is countable, since for each s, $\{F_s(x) : F'_s(x) = 0\}$ is finite. But F is continuous, so $F(I_n) = I_n$, so $F(I_n \setminus Z)$ has measure $\mu(I_n)$, and F fails to satisfy Lusin's (N). □

Recall that $F \in ACG$ (respectively ACG_*) if and only if F is in VBG (respectively VBG_*) and F satisfies (N). We do not know the precise descriptive complexity of the condition (N), so for the purposes of this analysis, it is easier to use a related and strictly stronger condition, Banach's condition (S).

Definition 12. *A function $F : I \to \mathbb{R}$ satisfies (S) if for every ε there is a δ such that for every set $A \subseteq I$ of Lebesgue measure less than δ, its image $F(A)$ has measure less than ε.*

In case F is continuous, condition (S) has a Π^0_3 equivalent definition similar to the definition of absolute continuity. For the rest of this section we adopt the convention that if $b < a$, $[a, b]$ will denote the interval $[b, a]$.

Definition 13. *A function $F : I \to \mathbb{R}$ satisfies* interval-(S) *if for every ε there is a δ such that for all non-decreasing sequences $a_0, b_0, \dots a_k, b_k$, if $\sum_i (b_i - a_i) < \delta$ and the intervals $[F(a_i), F(b_i)]$ are disjoint, then $\sum_i |F(b_i) - F(a_i)| \leq \varepsilon$.*

Proposition 25. *A function $F \in C(I)$ satisfies (S) if and only if it satisfies interval-(S).*

Proof. If F satisfies (S), then by the continuity of F, we have that $[F(a_i), F(b_i)] \subseteq F([a_i, b_i])$. Therefore, F satisfies interval-(S) with the same witnesses that it uses to satisfy (S). On the other hand, suppose F satisfies interval-(S), and let ε be given. Let $\varepsilon_0 < \varepsilon$. Let δ be the witness that F satisfies (S) for ε_0, and let $A \subseteq I$ with $\mu(A) < \delta$. Then A can be covered by $\cup_{i<\omega}(a_i, b_i)$ where $\mu(\cup_{i<\omega}(a_i, b_i)) < \delta$ as well. It suffices to show that for each k and for each $\varepsilon' > 0$, that $\mu(B_k) < \varepsilon_0 + \varepsilon'$, where $B_k = \cup_{i<k} F([a_i, b_i])$. By continuity of F, B_k is a finite union of intervals. Let $(c_j, d_j)_{j<\ell}$ be chosen so that

- For each j, there is an i such that $[c_j, d_j] \subseteq F([a_i, b_i])$,
- The $[c_j, d_j]$ are disjoint, and

- $\mu(B_k \setminus \cup_{j<\ell}[c_j, d_j]) < \varepsilon'$.

For each j, choose a'_j and b'_j so that for some i, $a'_j, b'_j \in [a_i, b_i]$, $F([a'_j, b'_j]) = [c_j, d_j]$, and $F(\{a'_j, b'_j\}) = \{c_j, d_j\}$. Then the $(a'_j, b'_j)_{i<\ell}$ are set up as in the definition of interval-(S), so $\mu(\cup_{j<\ell}[c_j, d_j]) \leq \varepsilon$, so $\mu(B_k) < \varepsilon + \varepsilon'$. □

Proposition 26. *If F is ACG_*, then F fulfills condition (S).*

Proof. We make use of Banach's property (T_1). By definition, a function F satisfies Banach's property (T_1) if $\{x \in \mathbb{R} : F^{-1}(\{x\})$ is infinite$\}$ is null. It is known (cf. [Sak64, Thm IX.8.4, pg 284; Thm IX.6.3, pg 279]) that a continuous function satisfies (S) if and only if it satisfies (N) and (T_1), and that any $F \in VBG_*$ satisfies (T_1). Since any $F \in ACG$ satisfies (N), it follows that any $F \in ACG_*$ satisfies (N) and (T_1), and therefore (S). □

Proposition 27. *Let $\alpha < \omega_1^Y$ and $\alpha > 1$. Then $ACG_{*\alpha}$ is $\Sigma^0_{2\alpha}(Y)$.*

Proof. Recall that a continuous function F is absolutely continuous if and only if it is of bounded variation and satisfies property (N). Furthermore, for any closed set E, the functions F_E and $F_{E,*}$ have property (N) if F does, because their linear portions cannot contribute to a failure of (N). Therefore, if F has the property (N), then we have not only that $F \in ACG$ (resp. ACG_*) if and only if $F \in VBG$ (resp. VBG_*), but also that $|F|_{AC} = |F|_{VB}$ (resp. $|F|_{AC_*} = |F|_{VB_*}$) in this case. This is because in the derivation process, the functions F_E (respectively $F_{E,*}$) are absolutely continuous on an interval if and only if they are of bounded variation on that interval. Since (N) is also necessary for members of ACG and ACG_*, we have

$$ACG_\alpha = VBG_\alpha \cap \{F \in C(I) : F \text{ satisfies } (N)\}$$

and similarly for $ACG_{*\alpha}$. However, since the descriptive complexity of (N) is not known (the naive complexity is not Borel), we cannot use this. What we can use in the case of ACG_* is the Π^0_3 property interval-(S), which is implied by ACG_* and implies (N). Therefore,

$$ACG_{*\alpha} = VBG_{*\alpha} \cap \{F \in C(I) : F \text{ satisfies interval-}(S)\}$$

and a similar statement does NOT hold for ACG, as there are functions in ACG that do not satisfy (S). Since satisfying interval-(S) is only Π^0_3, this equality and Proposition 22 together complete the proof for all $\alpha > 1$. □

We conclude this section by summarizing the results into the main theorem mentioned in the introduction.

Theorem 28. *Let $Y \in 2^\omega$, let $1 < \alpha < \omega_1^Y$, and let $A_\alpha = VBG_{*\alpha}, VBG_\alpha$ or $ACG_{*\alpha}$. Then A_α is $\Sigma^0_{2\alpha}(Y)$, and for any $\Sigma^0_{2\alpha}(Y)$ set B, there is a Y-computable reduction from B to A_α. In particular, A_α is $\mathbf{\Sigma}^0_{2\alpha}$-complete, and if $\alpha < \omega_1^{CK}$, then A_α is $\Sigma^0_{2\alpha}$-complete.*

5. Questions

Letting $|T|$ denote the usual well-founded rank of a tree $T \in WF$, a proof of the following in slightly different language can be found in [GMS13, Propositions 2.12 & 2.15].

Theorem 29 ([GMS13])**.** *If $\alpha < \omega_1^{ck}$, then $\{T : |T| \leq \omega\alpha\}$ is $\Sigma^0_{2\alpha}$-complete.*

Therefore, the complexity of initial segments of the usual well-founded rank increases by two jumps for every increase of ω in the rank. By contrast, all of the natural ranks considered in this paper have an increased complexity of two jumps for every increase of 1 in the rank. We wonder if there are examples of natural Π^1_1 ranks that don't fit one of these two patterns.

Question 30. *Is there an example of a* Π^1_1 *set* A*, a natural* Π^1_1 *rank which decomposes it as* $A = \cup_{\alpha<\omega_1} A_\alpha$*, and ordinals* β_α *such that each* A_α *is* Γ_{β_α}*-complete (here* $\Gamma = \Sigma$ *or* Π*) such that one of the following holds:*

(1) $\beta_\alpha + i = \beta_{\alpha+\delta}$ *for some* $\delta > \omega$ *and* $i \leq 2$*, or*
(2) $\beta_\alpha + i = \beta_{\alpha+1}$ *for some* $i > 2$*.*

It is open whether ACG provides an example satisfying part (2) of the above. Naively, the derivation process producing the classical rank on ACG could require $i = 4$.

Question 31. *What are the exact descriptive complexities of the sets* ACG_α*?*

The difficulty associated with applying the analysis of this paper to ACG is related to the unknown complexity of Lusin's (N).

Question 32. *What is the descriptive complexity of* $\{F \in C(I) : F$ *satisfies Lusin's* $(N)\}$*?*

Finally, although $\{T \in WF : |T|_{ls} \leq \alpha\}$ and $\{T \in WF : |T| \leq \omega\alpha\}$ are both $\Sigma^0_{2\alpha}$-complete, and thus there is a reduction from one to the other that passes through the universal set $\{X : n_0 \in H^X_{2^a}\}$ for appropriate a, we are not aware of a natural reduction between them.

Question 33. *Give natural computable functions* f *and* g *such that* $f^{-1}(\{T : |T|_{ls} \leq \alpha\}) = \{T : |T| \leq \omega\alpha\}$ *and* $g^{-1}(\{T : |T| \leq \omega\alpha\}) = \{T : |T|_{ls} \leq \alpha\}$*.*

References

[AK00] C. J. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[DK91] Randall Dougherty and Alexander S. Kechris. The complexity of antidifferentiation. *Adv. Math.*, 88(2):145–169, 1991.

[GMS13] Noam Greenberg, Antonio Montalbán, and Theodore A. Slaman. Relative to any non-hyperarithmetic set. *J. Math. Log.*, 13(1):1250007, 26, 2013.

[KW86] Alexander S. Kechris and W. Hugh Woodin. Ranks of differentiable functions. *Mathematika*, 33(2):252–278 (1987), 1986.

[Lem87] Steffen Lempp. Hyperarithmetical index sets in recursion theory. *Trans. Amer. Math. Soc.*, 303(2):559–583, 1987.

[Sak64] Stanisł aw Saks. *Theory of the integral*. Second revised edition. English translation by L. C. Young. With two additional notes by Stefan Banach. Dover Publications, Inc., New York, 1964.

[Wal17] Sean Walsh. Definability aspects of the Denjoy integral. *Fund. Math.*, 237(1):1–29, 2017.

[Wes14] Linda Brown Westrick. A lightface analysis of the differentiability rank. *J. Symb. Log.*, 79(1):240–265, 2014.

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