Comparing subshifts in $\mathbb{Z}$ and $\mathbb{Z}^2$

Linda Brown Westrick
University of Connecticut, Storrs

March 26, 2015
ASL North American Annual Meeting
University of Illinois Urbana-Champaign

Outline

- Subshifts
- The 1D-2D discrepancy for shifts of finite type
- Candidates for a better one-dimensional analogy

Subshifts

Subshifts

Definition

A subshift is a subset of $2^{\mathbb{Z}}$ that is both

- topologically closed
- closed under the shift operation $\{x_i\}_{i\in\mathbb{Z}} \mapsto \{x_{i-1}\}_{i\in\mathbb{Z}}$

Subshifts

Definition

A subshift is a subset of $2^{\mathbb{Z}}$ that is both

- topologically closed
- closed under the shift operation $\{x_i\}_{i\in\mathbb{Z}} \mapsto \{x_{i-1}\}_{i\in\mathbb{Z}}$

- For any $F \subseteq 2^{<\mathbb{N}}$, the following is a subshift:

$$X_F = \{x \in 2^{\mathbb{N}} : \text{ No } \sigma \in F \text{ appears as a subword in } x\ \}$$

- In fact, subshifts are characterized by their set of forbidden strings.
- Variations: Replace 2 with any finite alphabet. Replace $\mathbb{Z}$ with $\mathbb{Z}^d$ where d is any positive integer, and close under all d possible shift operations. Restrict F to be finite, or c.e.

Examples

In one dimension:

- The full shift $2^{\mathbb{Z}}$.
- The golden mean shift X_F where $F = \{\mathtt{11}\}$.
- The S-gap subshifts are X_F, where $F = \{10^n1 : n \notin S\}$ for some S.

Examples

In one dimension:

- The full shift $2^{\mathbb{Z}}$.
- The golden mean shift X_F where $F = \{\texttt{11}\}$.
- The S-gap subshifts are X_F, where $F = \{10^n1 : n \notin S\}$ for some S.

In two dimensions:

- forbidding $\left\{ \begin{array}{cc} 1 & 0 \\ 0, & 1 \end{array} \right\}$ produces elements of $2^{\mathbb{Z}^2}$ where each column contains either all 1's or all 0's.

Examples

In one dimension:

- The full shift $2^{\mathbb{Z}}$.
- The golden mean shift X_F where $F = \{\texttt{11}\}$.
- The S-gap subshifts are X_F, where $F = \{10^n1 : n \notin S\}$ for some S.

In two dimensions:

- forbidding $\left\{ \begin{array}{cc} 1 & 0 \\ 0, & 1 \end{array} \right\}$ produces elements of $2^{\mathbb{Z}^2}$ where each column contains either all 1's or all 0's.

Definition

- If F is finite, X_F is called a *shift of finite type (SFT).*
- If F is c.e., X_F is called a Π^0_1 *shift.*

The 1D-2D discrepancy for SFTs

With respect to the invariants:

Entropy
Medvedev degree

Entropy

Definition

For $X \subseteq 2^{\mathbb{Z}^d}$ a subshift, the entropy of X is

$$h(X) = \lim_{n\to\infty} \frac{\log |\{x[-n:n] : x \in X\}|}{(2n+1)^d},$$

where $x[-n:n]$ is the d-dimensional box of radius n found at the center of x.

Entropy measures the exponential growth rate of possibilities for initial segments of a subshift. If the number of possibilities for strings of length l grows as 2^{sl}, the entropy is s.

This limit converges from above. Thus the entropy of a Π^0_1 subshift is a right-r.e. number.

Entropy for Shifts of Finite Type (SFTs)

Theorem (Classical)

The entropies of the one-dimensional SFTs are exactly the non-negative rational multiples of logarithms of Perron numbers.

Entropy for Shifts of Finite Type (SFTs)

Theorem (Classical)

The entropies of the one-dimensional SFTs are exactly the non-negative rational multiples of logarithms of Perron numbers.

Theorem (Hochman and Meyerovitch 2010)

The entropies of the d-dimensional SFTs for $d > 1$ are exactly the right-r.e. numbers.

Medvedev Degree for SFTs

Definition

For $A, B \subseteq 2^\omega$, A is Medvedev reducible to B if there is a Turing functional Γ such that whenever $x \in B$, $\Gamma(x) \in A$.

Medvedev Degree for SFTs

Definition

For $A, B \subseteq 2^\omega$, A is Medvedev reducible to B if there is a Turing functional Γ such that whenever $x \in B$, $\Gamma(x) \in A$.

Fact (Classical)

Every one-dimensional SFT contains a periodic element. Therefore, all the one-dimensional SFTs have Medvedev degree $\mathbf{0}$.

Medvedev Degree for SFTs

Definition

For $A, B \subseteq 2^\omega$, A is Medvedev reducible to B if there is a Turing functional Γ such that whenever $x \in B$, $\Gamma(x) \in A$.

Fact (Classical)

Every one-dimensional SFT contains a periodic element. Therefore, all the one-dimensional SFTs have Medvedev degree $\mathbf{0}$.

Theorem (Simpson 2007)

Every Medvedev degree that contains a Π^0_1 set contains a two-dimensional SFT.

The 1D-2D discrepancy for SFTs

	$\mathbb{Z}$-SFT	$\mathbb{Z}^2$-SFT
Entropies	Perron-related reals	Right-r.e. reals
Medvedev degrees	**0** only	All possible

The 1D-2D discrepancy for SFTs

	$\mathbb{Z}$-SFT	$\mathbb{Z}^2$-SFT
Entropies	Perron-related reals	Right-r.e. reals
Medvedev degrees	$\mathbf{0}$ only	All possible

SFTs in one and $d > 1$ dimensions are not very similar to each other.

The 1D-2D discrepancy for SFTs

	$\mathbb{Z}$-SFT	$\mathbb{Z}^2$-SFT
Entropies	Perron-related reals	Right-r.e. reals
Medvedev degrees	**0** only	All possible

SFTs in one and $d > 1$ dimensions are not very similar to each other.

Perhaps there is some class of $\mathbb{Z}$-subshifts which are more similar to $\mathbb{Z}^2$-SFTs than $\mathbb{Z}$-SFTs are.

Why look for this?

Definition

Let X be a subshift.

- A *factor* of X is the image of X under a continuous, shift-invariant map.
- A factor of an SFT is called a *sofic shift*.

In general, if f is continuous and shift-invariant, $h(f(X)) \leq h(X)$.

Theorem (Coven & Paul 1975)

Every one-dimensional sofic shift is a factor of an SFT with the same entropy.

Open Question

Is every two-dimensional sofic shift a factor of an SFT with the same entropy?

What makes two dimensions different?

Extra space to move

Ability to embed a Turing machine

What makes two dimensions different?

Extra space to move

Ability to embed a Turing machine

- Both theorems about $\mathbb{Z}^2$-SFTs crucially use the fact that $\mathbb{Z}^2$-SFTs can embed a Turing machine.
- Every legal element of the SFT encodes a run of a Turing machine, which takes as its inputs the symbols of the SFT.
- (Actually the runs of an army of Turing machines!)
- If a Turing machine finds an undesirable pattern, it halts.
- The SFT forbids the pattern of a halted Turing machine, effectively forbidding the undesirable pattern.

Candidates for a better analogy

Two candidates:

Π_1^0 subshifts
Decidable subshifts

An additional invariant:
Effective dimension spectrum

Π_1^0 subshifts

Theorem (e.g. Hertling-Spandl 2008)

The entropies of the Π_1^0 subshifts are exactly the right-r.e. numbers.

Theorem (J. Miller 2012)

Every Medvedev degree that contains a Π_1^0 set contains a one-dimensional Π_1^0 subshift.

	$\mathbb{Z}$-SFT	$\mathbb{Z} - \Pi_1^0$ shift	$\mathbb{Z}^2$-SFT
Entropies	Perron-related	Right-r.e.	Right-r.e.
Medvedev degrees	**0** only	All possible	All possible

Embedding Π^0_1 shifts in $\mathbb{Z}^2$-SFTs

Theorem (Durand, Romashchenko & Shen 2012; independently Aubrun & Sablik 2013)

Every one-dimensional Π^0_1 subshift can be simulated inside a $\mathbb{Z}^2$-SFT.

- This improved a result of Hochman, who simulated them inside a $\mathbb{Z}^3$-SFT.
- The simulated subshift is encoded in the columns of the $\mathbb{Z}^2$-SFT.
- The main difficulty is coordinating the army of Turing machines to check every detail.

Complex Sequences

Definition

The effective dimension of $x \in 2^{\mathbb{Z}}$ is

$$\dim(x) = \liminf_{n\to\infty} \frac{K(x[-n:n])}{2n+1}.$$

Complex Sequences

Definition

The effective dimension of $x \in 2^{\mathbb{Z}}$ is

$$\dim(x) = \liminf_{n\to\infty} \frac{K(x[-n:n])}{2n+1}.$$

Definition

The (d, b)-shift-complex shift is the subshift X_F which forbids

$$F = \{\sigma : K(\sigma) < d|\sigma| - b\}.$$

- (For large enough b), X_F is a non-empty Π^0_1 subshift.
- Every element of X_F has effective dimension at least d.

Complex Tilings

Definition

The effective dimension of $x \in 2^{\mathbb{Z}^2}$ is

$$\dim(x) = \liminf_{n \to \infty} \frac{K(x[-n:n])}{(2n+1)^2}.$$

Complex Tilings

Definition

The effective dimension of $x \in 2^{\mathbb{Z}^2}$ is

$$\dim(x) = \liminf_{n\to\infty} \frac{K(x[-n:n])}{(2n+1)^2}.$$

Theorem (Durand, Levin & Shen 2006)

Every $\mathbb{Z}^2$-SFT contains an element x for which $K(x[-n:n])$ grows linearly with n (and this bound is tight).

Therefore, every $\mathbb{Z}^2$-SFT contains an element of effective dimension 0.

Effective dimension spectrum

Definition

The effective dimension spectrum of a set X is $\{\dim x : x \in X\}$.

Theorem (Simpson 2011)

Let X be a subshift in any number of dimensions. The effective dimension spectrum of X has a maximum element, which is the entropy $h(X)$.

Effective dimension spectrum

Definition

The effective dimension spectrum of a set X is $\{\dim x : x \in X\}$.

Theorem (Simpson 2011)

Let X be a subshift in any number of dimensions. The effective dimension spectrum of X has a maximum element, which is the entropy $h(X)$.

Theorem (W.)

Let X be an SFT in any number of dimensions. The effective dimension spectrum of X is $[0, h(X)]$.

Open problem: characterize the effective dimension spectra of Π^0_1 subshifts. But it is known that more variety is possible, e.g. shift-complex subshifts.

The Π^0_1 analogy breaks

	$\mathbb{Z}$-SFT	$\mathbb{Z} - \Pi^0_1$ shift	$\mathbb{Z}^2$-SFT
Entropies	Perron-related	Right-r.e.	Right-r.e.
Medvedev degrees	**0** only	All possible	All possible
Dimension Spectrum	$[0, h]$	variety	$[0, h]$

The Π^0_1 analogy breaks

	$\mathbb{Z}$-SFT	$\mathbb{Z} - \Pi^0_1$ shift	$\mathbb{Z}^2$-SFT
Entropies	Perron-related	Right-r.e.	Right-r.e.
Medvedev degrees	**0** only	All possible	All possible
Dimension Spectrum	$[0, h]$	variety	$[0, h]$

- The one-dimensional Π^0_1 subshifts are too powerful because their (external) Turing machine has unlimited access to the input.
- The Turing machines of d-dimensional SFTs use one dimension of the subshift as "time"; therefore, they can only look at a $(d-1)$-dimensional portion of their input.

Decidable subshifts

Definition

A subshift X is decidable if for every σ, it is decidable whether σ can be extended to some $x \in X$.

Decidable subshifts are Π^0_1, but not vice versa.

Decidable subshifts

Definition

A subshift X is decidable if for every σ, it is decidable whether σ can be extended to some $x \in X$.

Decidable subshifts are Π^0_1, but not vice versa.

Theorem (Hertling-Spandl 2008)

The entropies of the one-dimensional decidable subshifts are exactly the right-r.e. numbers.

Decidable subshifts

Definition

A subshift X is decidable if for every σ, it is decidable whether σ can be extended to some $x \in X$.

Decidable subshifts are Π^0_1, but not vice versa.

Theorem (Hertling-Spandl 2008)

The entropies of the one-dimensional decidable subshifts are exactly the right-r.e. numbers.

However, every decidable subshift contains a computable element, so has Medvedev degree 0.

Decidable subshifts

Definition

A subshift X is decidable if for every σ, it is decidable whether σ can be extended to some $x \in X$.

Decidable subshifts are Π^0_1, but not vice versa.

Theorem (Hertling-Spandl 2008)

The entropies of the one-dimensional decidable subshifts are exactly the right-r.e. numbers.

However, every decidable subshift contains a computable element, so has Medvedev degree 0.

Theorem (W.)

There is a one-dimensional decidable subshift whose dimension spectrum contains 0 *and* $1/3$*, but does not contain the interval* $(0, 1/3)$.

Summary

	$\mathbb{Z}$-SFT	$\mathbb{Z} - \Pi^0_1$ shift	$\mathbb{Z}$-decidable	$\mathbb{Z}^2$-SFT
Entropies	Perron-related	Right-r.e.	Right-r.e.	Right-r.e.
Medvedev	**0** only	All possible	**0** only	All possible
Spectrum	$[0, h]$	variety	variety	$[0, h]$

Parting questions:

- Is there a class of one-dimensional subshifts which behaves like a higher-dimensional SFT?
- If not (or to the extent not), why?