A Lightface Analysis of the Differentiability Rank

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Computability Across Mathematics @ AMS St. Louis

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Differentiability

Let $D = \{f \in C[0,1] : f \text{ is differentiable}\}$.

$$f \in D \iff \forall x \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \text{ exists} \right)$$
$$\forall x (\forall \epsilon \exists \delta \forall h (|h| < \delta \rightarrow \dots))$$

D is Π^1_1.

Mazurkiewicz, 1936

$\{f : f \text{ is differentiable}\}$ is Π^1_1-complete.

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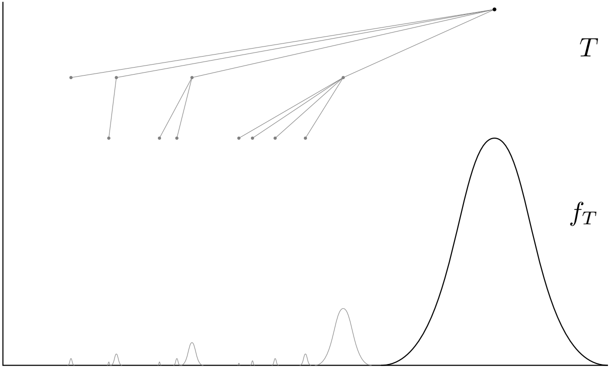

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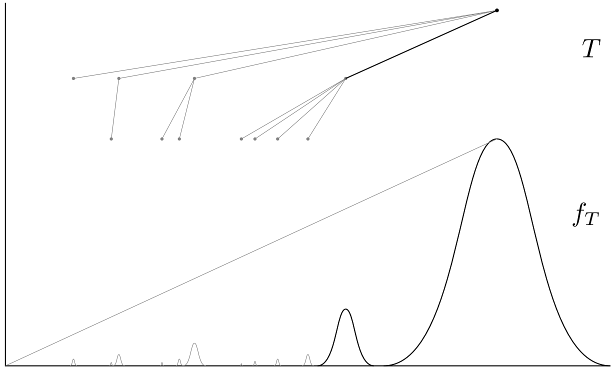

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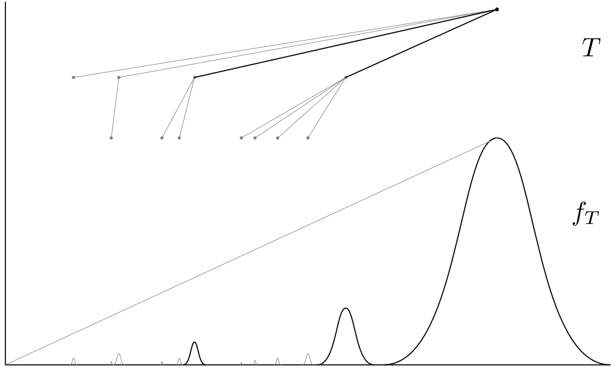

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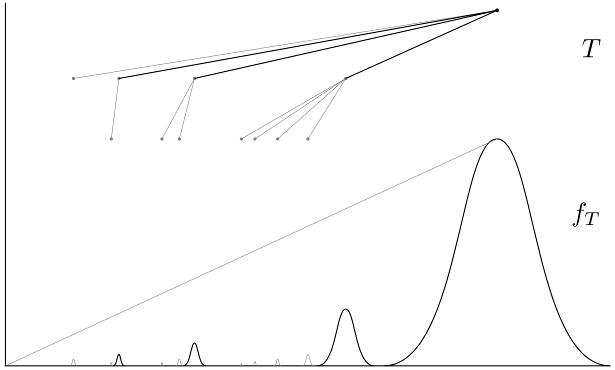

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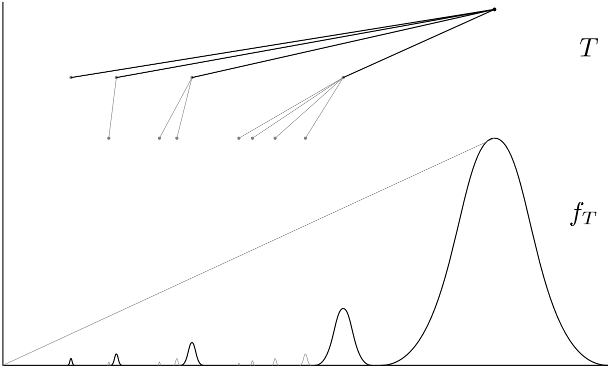

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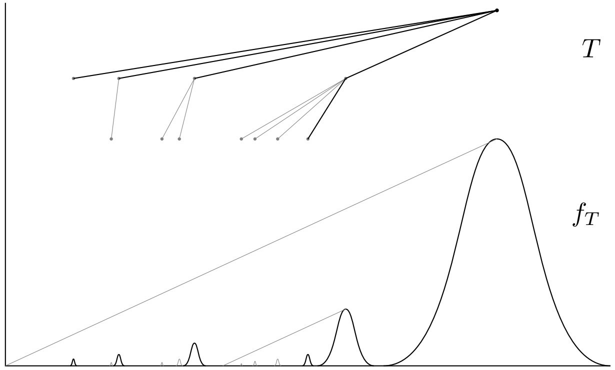

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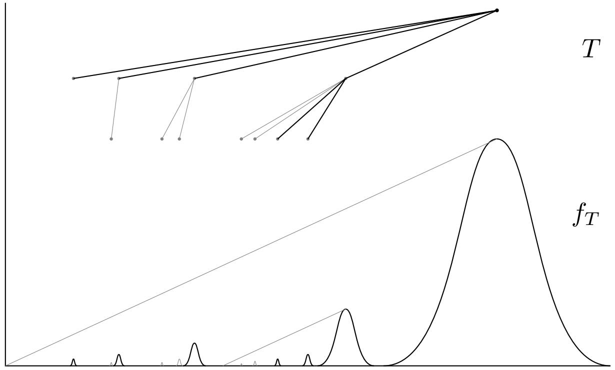

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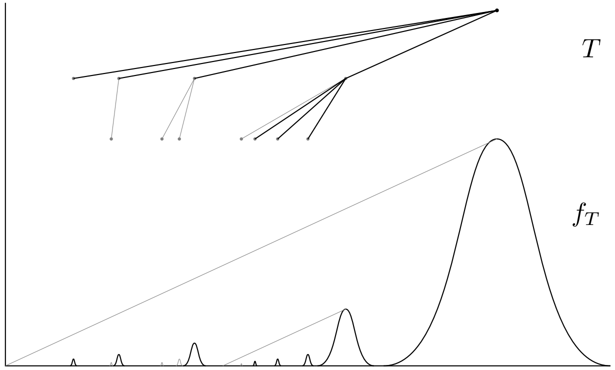

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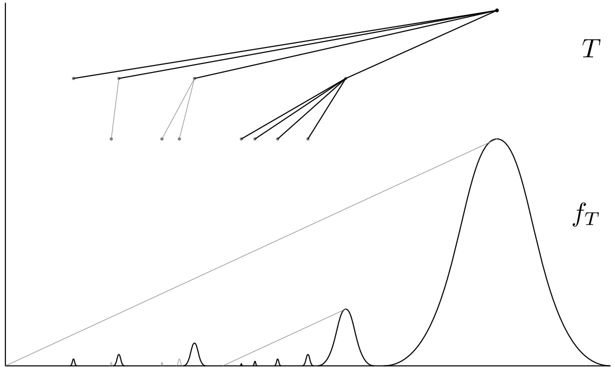

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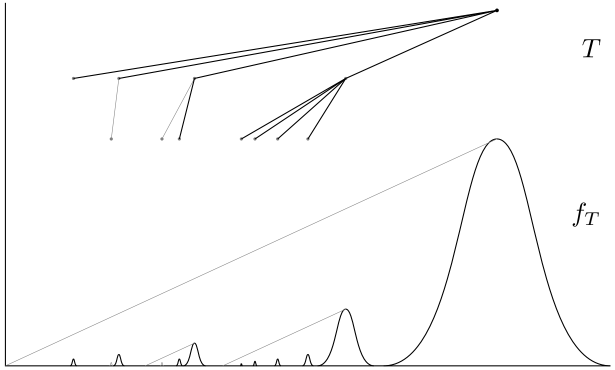

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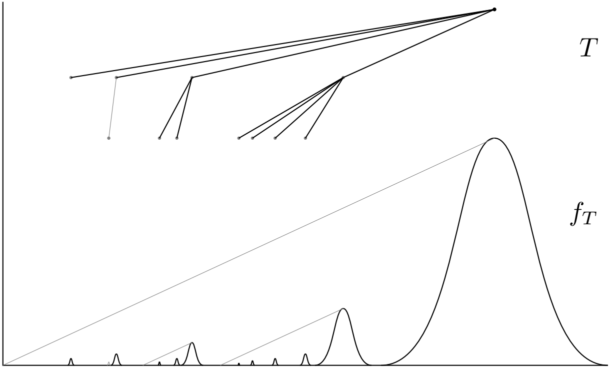

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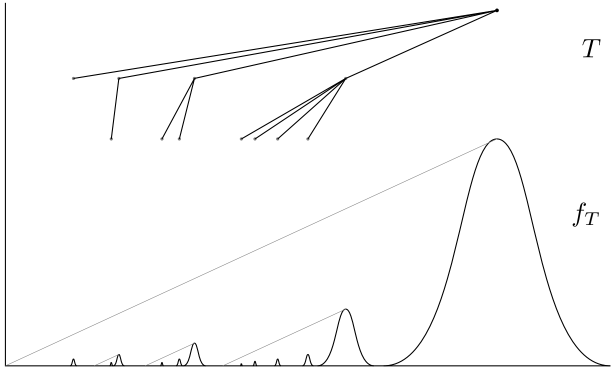

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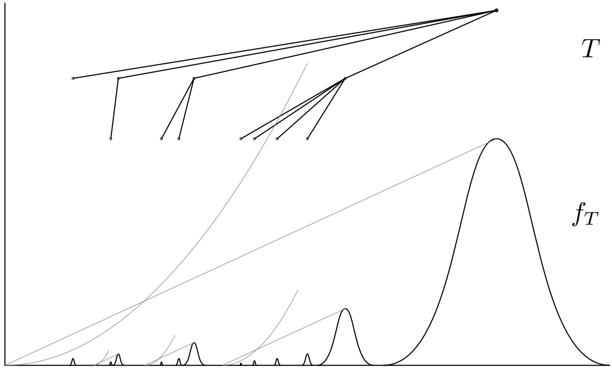

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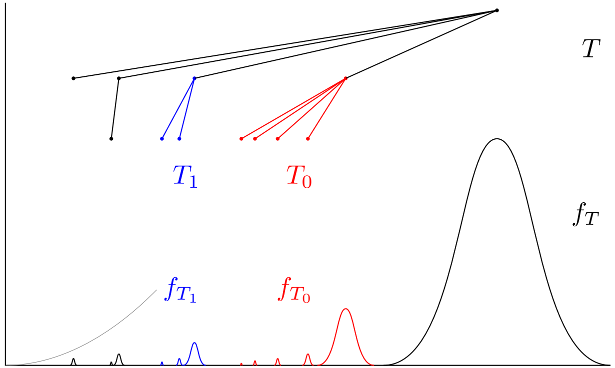

Suppose T is well-founded and each f_{T_n} is differentiable.

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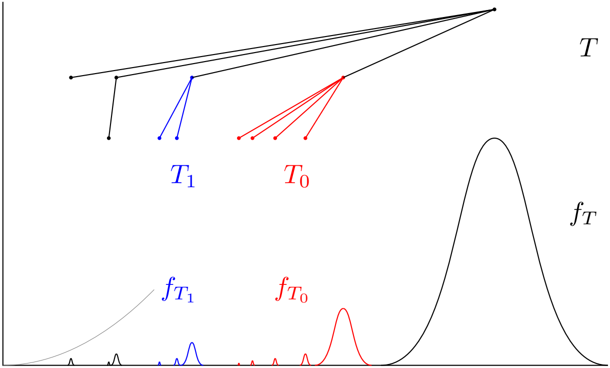

Suppose T is well-founded and each f_{T_n} is differentiable.

Then f_T is differentiable.

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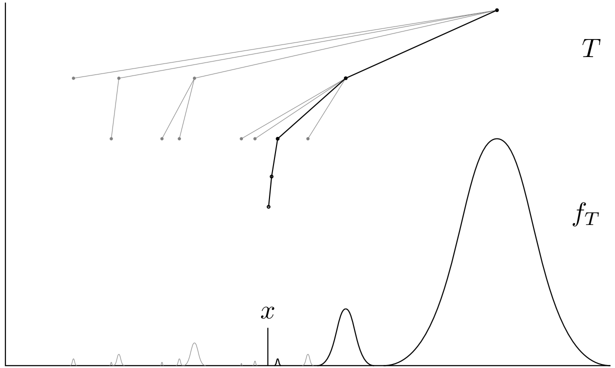

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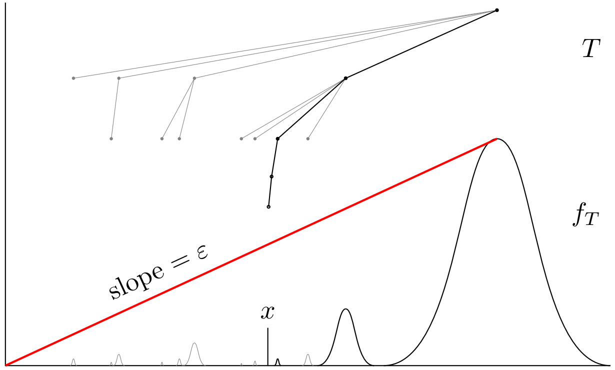

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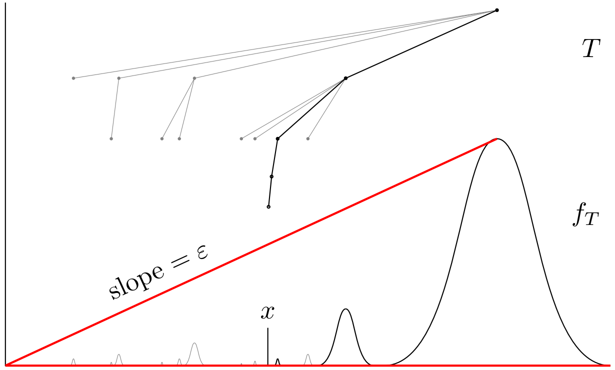

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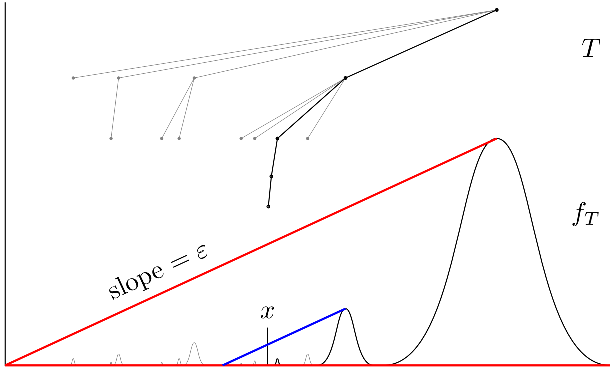

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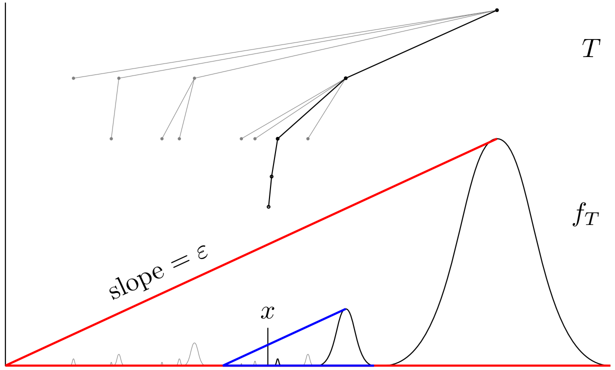

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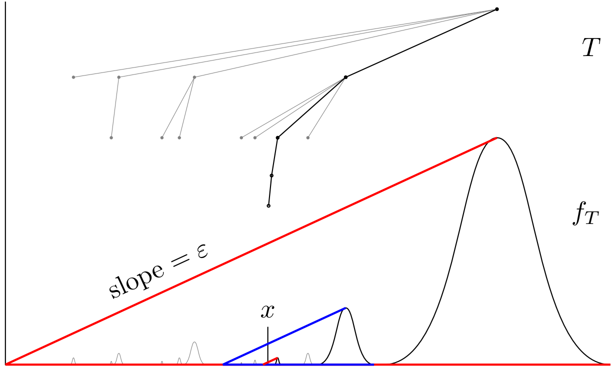

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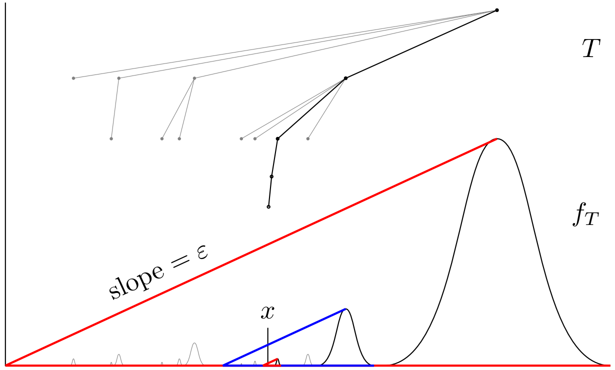

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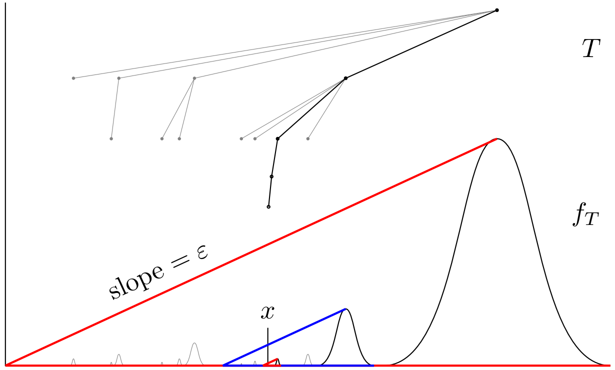

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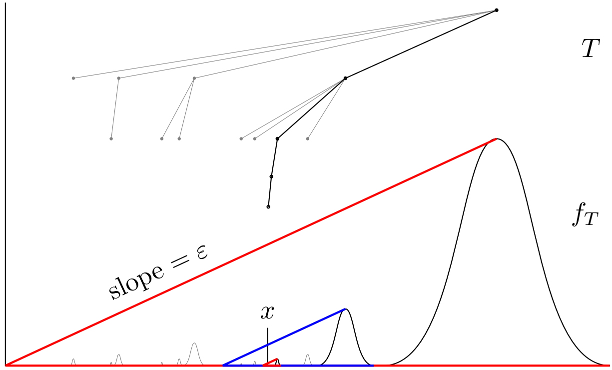

Suppose T is ill-founded.

Then f_T is not differentiable.
In fact, the oscillation of $\frac{f_T(x+h)-f_T(x)}{h}$ is ε.

ε-differentiability

- Let D_ε denote the functions f for which $|D_- f(x) - D^- f(x)| \leq \varepsilon$.
- We wanted:
$$(WF, \neg WF) \to (D, \neg D).$$
- We got a stronger result:
$$(WF, \neg WF) \to (D, \neg D_\varepsilon).$$
- Everything that is hard about D shows up in D_ε.
- Things are nicer in D_ε, so we restrict our attention there.
- Note:
$$D = \bigcap_{\varepsilon > 0} D_\varepsilon.$$
- By definition D_ε is Π^1_1. The stronger result showed D_ε is Π^1_1-complete.

A rank on D_ε functions

Given f, Kechris and Woodin defined a transfinite sequence of closed sets which eventually become empty if and only if $f \in D_\varepsilon$.

Definition (Kechris-Woodin 1986)

$$P^0 = [0,1]$$
$$P^{\alpha+1} = \left\{ x \in P^\alpha : \text{ conditions } \right\}$$
$$P^\lambda = \cap_{\alpha<\lambda} P^\alpha$$

Definition

If $f \in D_\varepsilon$, then $|f|_\varepsilon$ denotes the least α for which $P^\alpha = \emptyset$.

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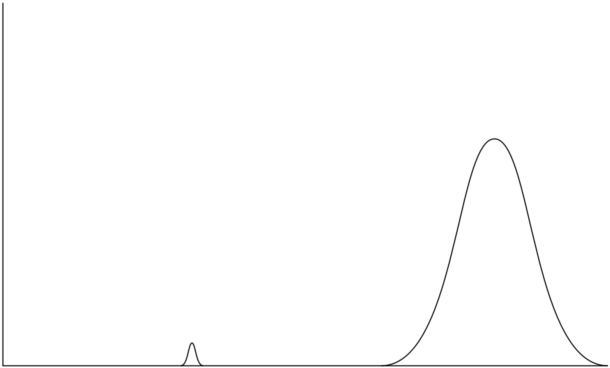

A rank 1 function.
$P^0 = [0,1]$
$P^1 = \emptyset$
$|f|_\varepsilon = 1$

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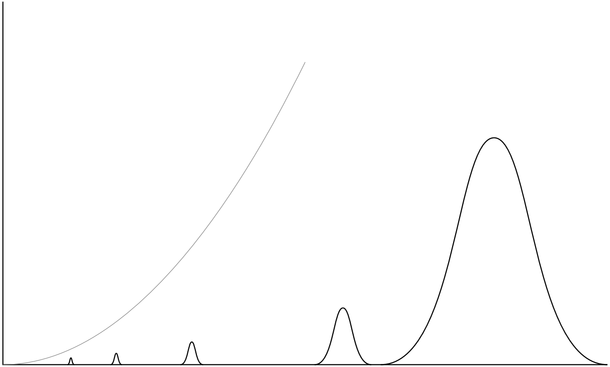

A rank 2 function.

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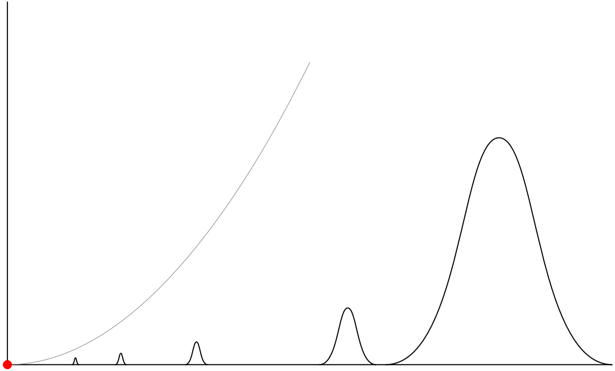

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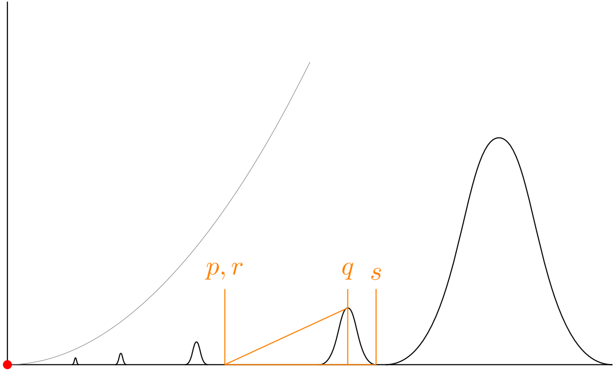

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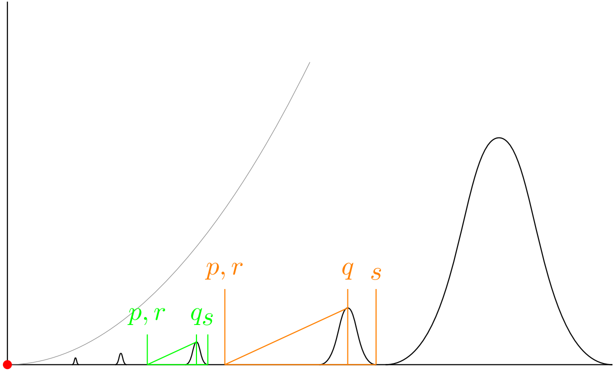

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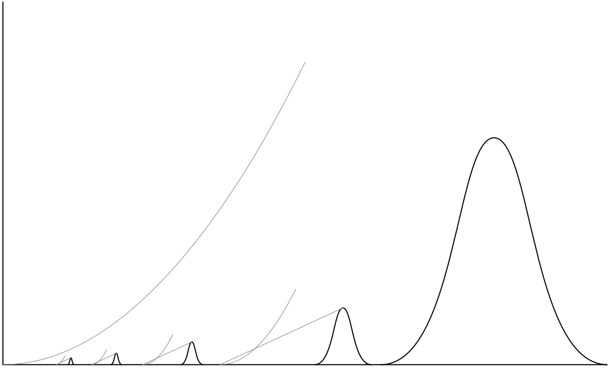

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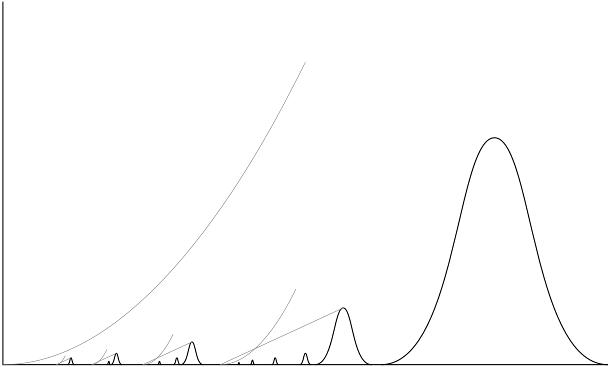

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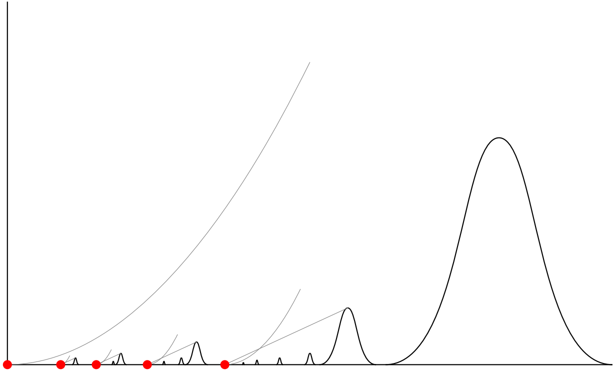

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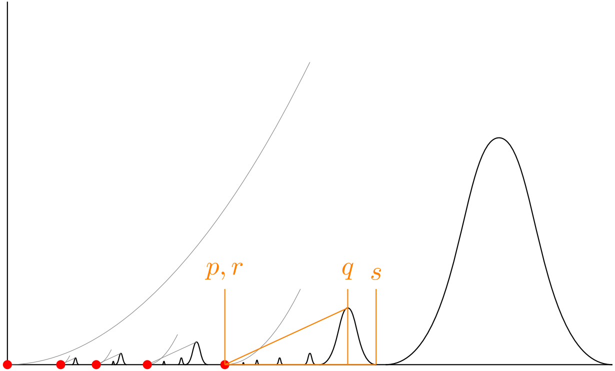

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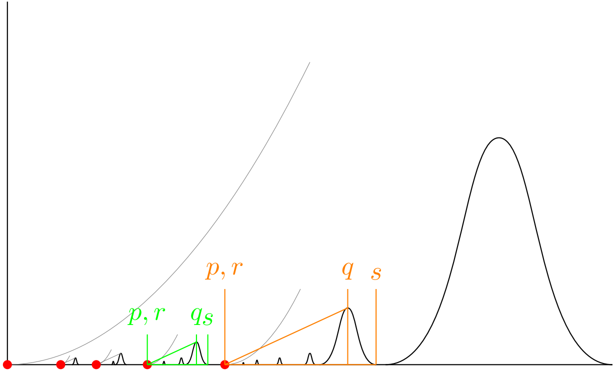

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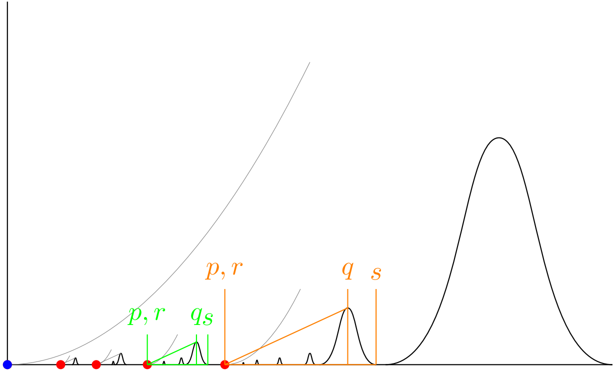
$|f|_\varepsilon = 3$

A rank on D_ε functions

$$P^{\alpha+1} = \left\{ x \in P^\alpha : \begin{array}{l} \text{For all open intervals } I \ni x, \text{there are } p,q,r,s \in I \text{ s.t.} \\ \left| \frac{f(p)-f(q)}{p-q} - \frac{f(r)-f(s)}{r-s} \right| > \varepsilon \text{ and } [p,q] \cap [r,s] \cap P^\alpha \neq \emptyset \end{array} \right\}$$



A rank 3 function.

$P^0 = [0,1]$

P^1

$P^2 = \{0\}$

$P^3 = \emptyset$

$|f|_\varepsilon = 3$

Our Goal

Definition (Ash-Knight, 2000)

For infinite α, X is Σ_α if $X \equiv_1 H_{2^a}$ for any a with $|a|_\mathcal{O} = \alpha$.

Remark: Here $(\emptyset^{(\omega)})'$ is a Σ_ω-complete set.

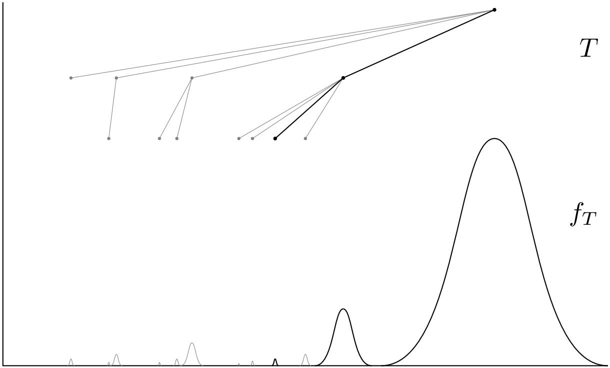
Theorem (W)

The set $\{f : |f|_\varepsilon < \alpha + 1\}$ is $\Sigma_{2\alpha}$-complete for any constructive ordinal $\alpha > 0$.

- Easy direction: $\{f : |f|_\varepsilon < \alpha + 1\}$ is $\Sigma_{2\alpha}$.
- Hard direction: Requires a reduction $(\Sigma_{2\alpha}, \Pi_{2\alpha}) \to (|f|_\varepsilon \leq \alpha, |f|_\varepsilon > \alpha)$.

A finer analysis

We return to the $(WF, \neg WF) \to (D, \neg D_\varepsilon)$ reduction.
If T is WF, then $|f_T|_\varepsilon$ exists.
But $|f_T|_\varepsilon$ is not, in general, equal to the well-founded rank of T.



The well-founded rank of T is 3, but $|f_T|_\varepsilon = 1$.

Differentiability

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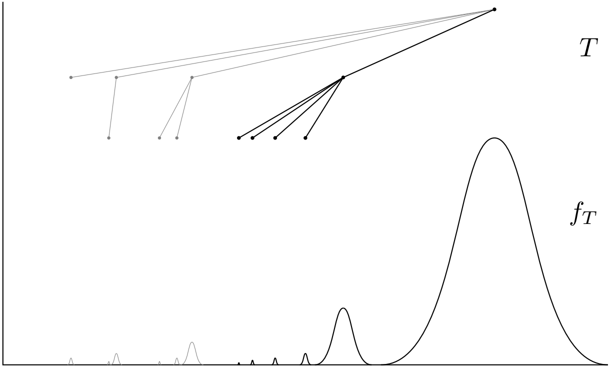


The well-founded rank of T is still 3, but now $|f_T|_\varepsilon = 2$.

How can one control the rank of f_T?

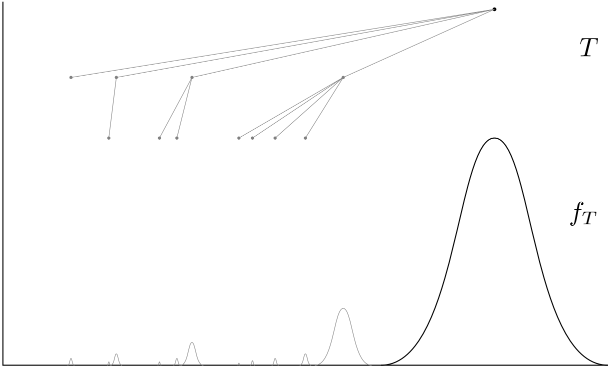
We will define an alternate rank on trees to answer this question.

Limsup rank

Definition

The **limsup rank** of a well-founded tree T is 0 if $T = \emptyset$ and

$$|T|_{ls} = \max(\sup_n |T_n|_{ls}, [\limsup_n |T_n|_{ls}] + 1) \text{ otherwise.}$$



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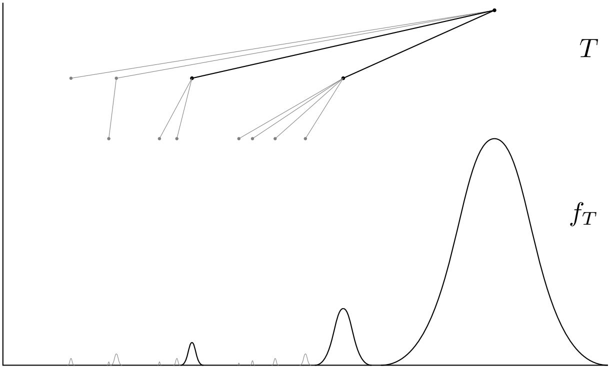

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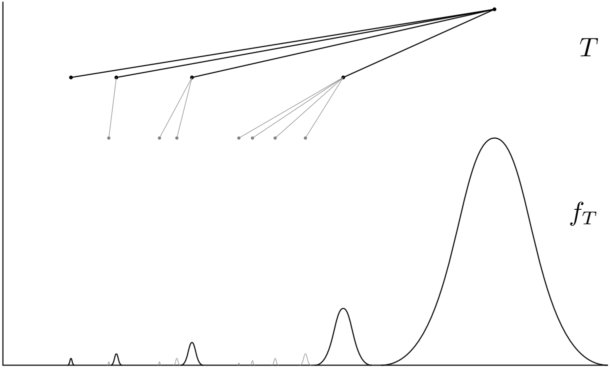

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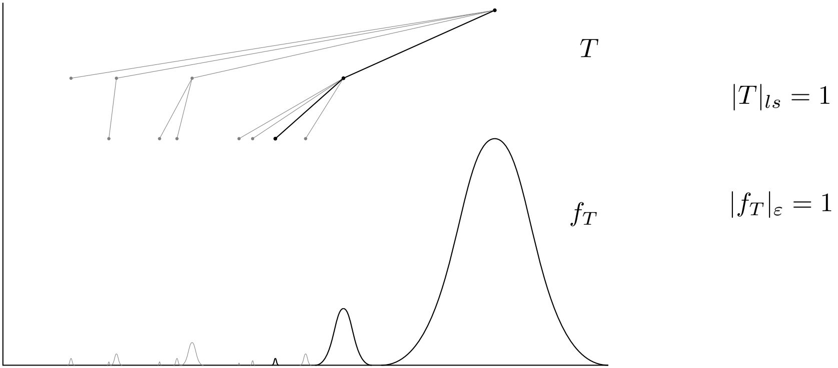

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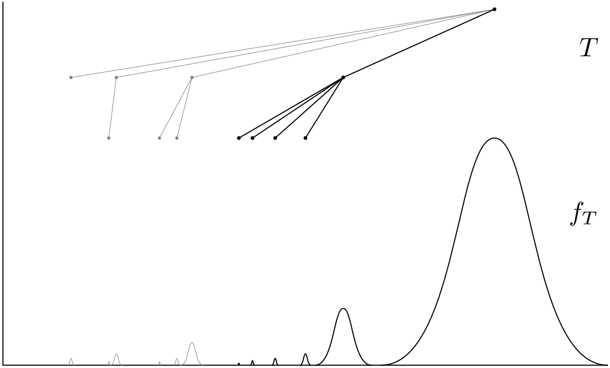

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Results

Lemma

For all well-founded nonempty T*, we have* $|T|_{ls} = |f_T|_\varepsilon$.

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Theorem (W)

Uniformly in a finite sequence of statements $P_1, \ldots, P_k$*, where each* P_i *is* $\Sigma_{2\alpha_i}$*, one may produce a tree* $T(P_1, \ldots, P_k)$ *such that*

$$|T|_{ls} = \begin{cases} \leq \alpha_i & \text{for each } i \text{ such that } P_i \text{ holds} \\ \max_i \alpha_i + 1 & \text{if all statements fail} \end{cases}$$

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Corollary

For constructive $\alpha > 0$, $\{f : |f|_\varepsilon < \alpha + 1\}$ is $\Sigma_{2\alpha}$-complete.

Cantor-Bendixson Rank

Corollary (Lempp 1987)

Let D^α denote the αth Cantor-Bendixson derivative. For each constructive $\alpha > 0$, the set $\{T \subseteq 2^{<\omega} : T$ has no dead ends and $D^\alpha(T) = \emptyset\}$ is $\Sigma_{2\alpha}$-complete.

These sets are naively $\Sigma_{2\alpha}$.
If $[S] \subseteq 2^{<\omega}$ is countable, let $|S|_{CB}$ denote the least α such that $D^\alpha S = \emptyset$.

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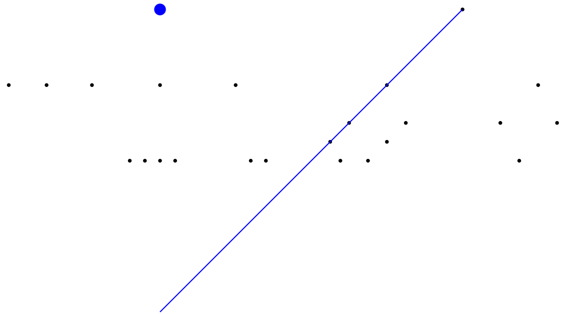

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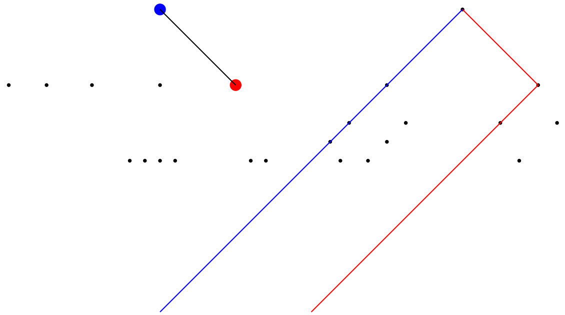

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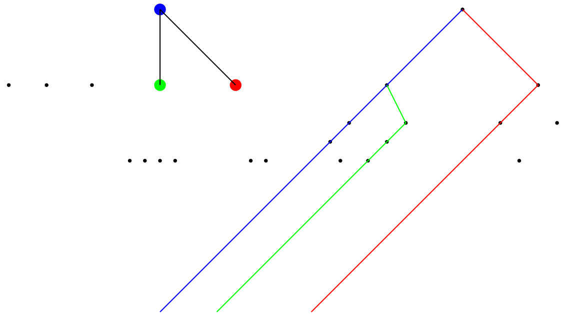

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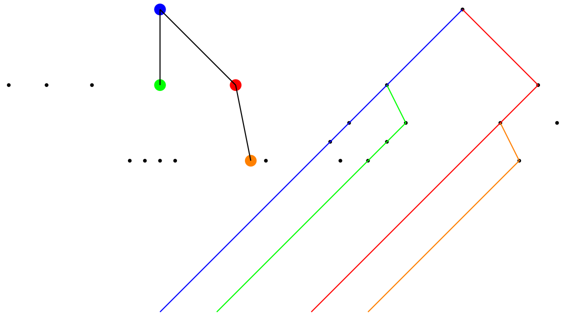

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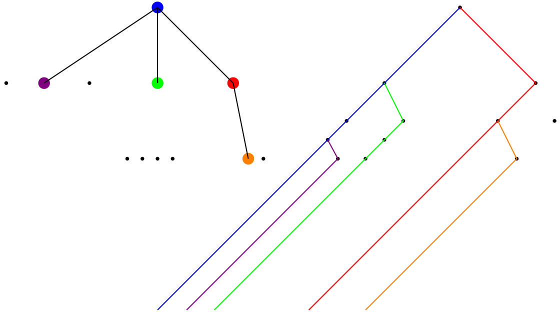

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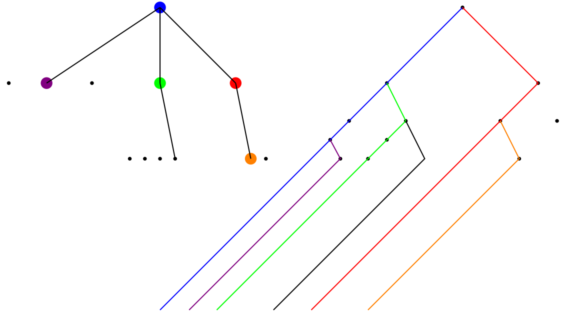

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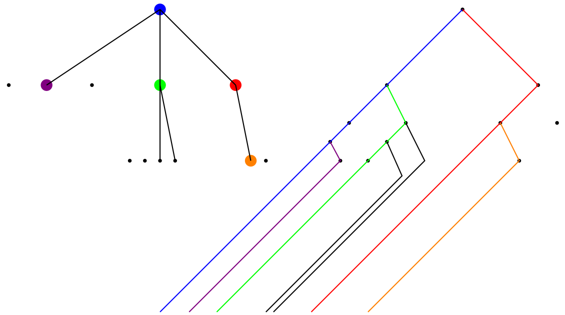

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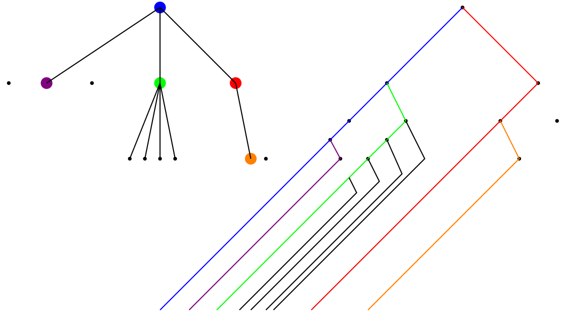

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Denjoy integration

Definition

A function $F \in C[0,1]$ is a Denjoy integral if $F(x) = \int_0^x f(x)dx$ for some measurable function f, where $\int$ refers to the transfinite process of Denjoy integration (not defined here).

Definition

A Denjoy integral F has rank α if the integration process that produced it takes α steps. We write $|F|_D = \alpha$.

One may define a computable reduction $T \mapsto F_T$ from WF to $C[0,1]$ such that for each $T \in WF$, $|T|_{ls} = |F_T|_D$.

Corollary

For each constructive $\alpha > 1$, $\{F : |F|_D < \alpha + 1\}$ *is* $\Sigma_{2\alpha}$*-complete.*

Unifying Principle

Each node in our trees T can be mapped to a "unit of badness" in a hierarchy classification problem.

- For differentiation, a unit of badness is a pair of disagreeing secants.
- For Cantor-Bendixson rank, a unit of badness is a path.
- For Denjoy integration, a unit of badness is an increase in total variation.