Homomesy in products of two chains

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Abstract

Many invertible actions \( \tau \) on a set \( S \) of combinatorial objects, along with a natural statistic \( f \) on \( S \), exhibit the following property which we dub homomesy: the average of \( f \) over each \( \tau \)-orbit in \( S \) is the same as the average of \( f \) over the whole set \( S \). This phenomenon was first noticed by Panyushev in 2007 in the context of the rowmotion map acting on the set of antichains of a root poset; Armstrong, Stump, and Thomas proved Panyushev’s conjecture in 2011. We describe a theoretical framework for results of this kind that seems to apply more broadly, giving examples in a variety of contexts. These include linear actions on vector spaces, sandpile dynamics, Suter’s action on certain subposets of Young’s Lattice, promotion of rectangular semi-standard Young tableaux, and the rowmotion and promotion operators acting on certain posets. We give a detailed description of the latter situation for products of two chains.

Keywords: antichains, ballot theorems, homomesy, Lyness 5-cycle, orbit, order ideals, Panyushev complementation, permutations, poset, product of chains, promotion, rowmotion, sandpile, Suter’s symmetry, toggle group, Young’s Lattice, Young tableaux.

1 Introduction

We begin with the definition of our main unifying concept, and supporting nomenclature.

Definition 1. Given a set \( S \), an invertible map \( \tau \) from \( S \) to itself such that each \( \tau \)-orbit is finite, and a function (or “statistic”) \( f : S \to K \) taking values in some field \( K \) of characteristic zero, we say the triple \((S, \tau, f)\) exhibits homomesy\(^1\) iff there exists a constant \( c \in K \) such that for every \( \tau \)-orbit \( \mathcal{O} \subset S \)

\[
\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c. \tag{1}
\]

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\(^1\)Greek for “same middle”
In this situation we say that the function \( f : S \rightarrow K \) is \textit{homomesic} under the action of \( \tau \) on \( S \), or more specifically \textit{c-mesic}.

When \( S \) is a finite set, homomesy can be restated equivalently as all orbit-averages being equal to the global average:

\[
\frac{1}{\#O} \sum_{x \in O} f(x) = \frac{1}{\#S} \sum_{x \in S} f(x). \tag{2}
\]

We will also apply the term homomesy more broadly to include the case that the statistic \( f \) takes values in a vector space over a field of characteristic 0 (as in sections 2.4 and 2.7).

We have found many instances of (2) where \( S \) is a finite collection of combinatorial objects (e.g., order ideals in a poset), \( \tau \) is a natural action on \( S \) (e.g., rowmotion or promotion), and \( f \) is a natural measure on \( S \) (e.g., cardinality). Many (but far from all) situations that support examples of homomesy also support examples of the cyclic sieving phenomenon of Reiner, Stanton, and White [RSW04], and more exploration of the links and differences is certainly in order. At the stated level of generality this notion appears to be new, but specific instances can be found in earlier literature. In particular, Panyushev [Pan09] conjectured and Armstrong, Stump, and Thomas [AST11] proved the following homomesy: if \( S \) is the set of antichains in the root poset of a finite Weyl group, \( \Phi \) is the operation variously called the Brouwer-Schrijver map [BS74], the Fon-der-Flaass map [Fon93, CF95], the reverse map [Pan09], Panyushev complementation [AST11], and rowmotion [SW12], and \( f(A) \) is the cardinality of the antichain \( A \), then \((S, \Phi, f)\) satisfies (2).

Our main results for this paper involve studying the action of this rowmotion operator and also the (Striker-Williams) promotion operator associated with the poset \( P = [a] \times [b] \). (See Section 3 for precise definitions. Note that we use \([n]\) to denote both the set \( \{1, \ldots, n\} \) and the natural poset with those elements, according to context.) We show that the statistic \( f := \#A \), the size of the antichain, is homomesic with respect to the promotion operator, and that the statistic \( f = \#I(A) \), the size of the corresponding order ideal, is homomesic with respect to both the promotion and rowmotion operators.

Although these results are of intrinsic interest, we think the main contribution of the paper is its identification of homomesy as a phenomenon that occurs quite widely. Within any linear space of functions on \( S \), the functions that are 0-mesic under \( \tau \), like the functions that are invariant under \( \tau \), form a subspace. There is a loose sense in which the notions of invariance and homomesy (or, more strictly speaking, 0-mesy) are complementary; an extremely clean case of this complementarity is outlined in subsection 2.4, and a related complementarity (in the context of continuous rather than discrete orbits) is sketched in subsection 2.5. This article gives a general overview of the broader picture as well as a few specific examples done in more detail for the operators of promotion and rowmotion associated with the poset \([a] \times [b]\).

We provide examples of homomesy in a wide variety of contexts. These include the following actions with corresponding statistics, each of which is explained in more detail in the indicated subsections. All the examples in Section 2 are fairly independent of each other and of our main new results in Section 3, so the reader may focus on some examples more than others, according to taste.

1. reversal of permutations with the statistic that counts inversions \([\S 2.1]\);
2. cyclic rotation of words on $\{-1,+1\}$ with the $\{0,1\}$-function that indicates whether a word satisfies the ballot condition [§ 2.2];

3. cyclic rotation of words on $\{-1,+1\}$ with the statistic that counts the number of (multiset) inversions in the word [§ 2.3];

4. linear maps which satisfy $T^n = 1$ acting in a vector space $V$ with statistic the identity function [§ 2.4];

5. circular actions (given by trigonometric functions) on the vector space $\mathbb{R}[x,y]$ with statistic given by the time-average around the circle [§ 2.5];

6. the Lyness 5-cycle acting on (most of) $\mathbb{R}^2$ with $f((x,y)) = \log|h(x)|$ as the statistic [§ 2.6];

7. the action on recurrent sandpile configurations given by adding 1 grain to the source vertex then allowing the system to stabilize, with statistic the firing vector [§ 2.7];

8. Suter’s action on Young diagrams with a weighted cardinality statistic [§ 2.8];

9. promotion (in the sense of Schützenberger) acting on semistandard Young tableaux of rectangular shape with statistic given by summing the entries in any centrally-symmetric subset of cells of the tableaux [§ 2.9];

10. promotion (in the sense of [SW12]) acting on the set of order ideals of $[a] \times [b]$ with the cardinality statistic [§ 3.2];

11. rowmotion acting on the set of order ideals of $[a] \times [b]$ with the cardinality statistic [§ 3.3.1]; and

12. rowmotion acting on the set of antichains of $[a] \times [b]$ with the cardinality statistic [§ 3.3.2].

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2 Examples of Homomesy

Here we give a variety of examples of homomesy in combinatorics, the first two of which long predate the general notion of homomesy; we also give non-combinatorial examples that establish links with other branches of mathematics. For examples of homomesy associated with piecewise-linear maps and birational maps, see [EP13].

2.1 Inversions in permutations

Let $S$ be the set of permutations of $\{1, 2, \ldots, n\}$, let $\tau$ send $\pi_1 \pi_2 \ldots \pi_n$ (a permutation written in one-line notation) to its reversal $\pi_n \pi_{n-1} \ldots \pi_1$ and let $f(\pi)$ be the number of inversions in $\pi$. Since $\tau^2$ is the identity, and since $f(\pi) + f(\tau(\pi)) = n(n-1)/2$, $f$ is $c$-mesic under the action of $\tau$, where $c = n(n-1)/4$.

2.2 Ballot theorems

Fix two nonnegative integers $a$ and $b$ and set $n = a + b$. Let $S$ be the set of words $(s_1, s_2, \ldots, s_n)$ of length $n$, consisting of $a$ letters equal to $-1$ and $b$ letters equal to $+1$; we think of each such word as an order for counting $n$ ballots in a two-way election, $a$ of which are for candidate A and $b$ of which are for candidate B. If $a < b$, then candidate B will be deemed the winner once all $a+b$ ballots have been counted, and we ask for the probability that at every stage in the counting of the ballots, candidate B is in the lead. This probability is the same as the expected value of $f(s)$, where $f(s)$ is 1 if $s_1 + \cdots + s_i > 0$ for all $1 \leq i \leq n$ and is 0 otherwise, and where $s$ is chosen uniformly at random from $S$. Bertrand’s Theorem states that this probability is $(b-a)/(b+a)$.

Dvoretzky and Motzkin’s famous “cycle lemma” proof of Bertrand’s Theorem [DM47] may be recast in our framework as follows:

**Proposition 2.** Let $\tau := C_L : S \rightarrow S$ be the leftward cyclic shift operator that sends $(s_1, s_2, s_3, \ldots, s_n)$ to $(s_2, s_3, \ldots, s_n, s_1)$. Then over any orbit $O$ one has

$$\frac{1}{\#O} \sum_{s \in O} f(s) = \frac{b-a}{b+a}.$$ 

In other words, $f$ is $c$-mesic with $c = \frac{b-a}{b+a}$. See [R07] for details.

Dvoretzky and Motzkin [DM47] used this lemma to prove a more general version of Bertrand’s Theorem originally due to Barbier [B1887]: if $b > ra$ for some nonnegative integer $r$, then the probability that throughout the counting candidate B always has more than $r$ times as many votes as candidate A is $(b-r a)/(b+a)$.

2.3 Inversions in two-element multiset permutations

As in the preceding section, let $S$ be the set of words of length $n = a + b$ consisting of $a$ letters equal to $-1$ and $b$ letters equal to $+1$ (without the requirement that $a < b$), and let $f(s) := \text{inv}(s) := \#\{i < j : s_i > s_j\}$. For fixed $i < j$, the number of $s$ in $S$ with $s_i > s_j$ (i.e., with $s_i = 1$ and $s_j = -1$) is $\binom{n-2}{a-1}$, so the probability that an $s$ chosen uniformly at
random from \( S \) satisfies \( s_i > s_j \) is \((\binom{n-2}{i-1})/\binom{n}{i}\); hence by additivity of expectation, the expected value of \( \text{inv}(s) \) is \( \sum_{i<j} \frac{ab}{n(n-1)} = \frac{n(n-1)}{2} \frac{ab}{n(n-1)} = ab/2. \)

**Proposition 3.** Let \( \tau \) be the left-shift \( C_L \) on \( S \) and \( f(s) := \text{inv}(s) \) as above. Then over each orbit \( O \) we have

\[
\frac{1}{\#O} \sum_{s \in O} f(s) = \frac{ab}{2} = \frac{1}{\#S} \sum_{s \in S} f(s).
\]

In other words, the inversion statistic is homomesic under the action of cyclic rotation, with average \( ab/2 \) on each orbit.

One way to prove Proposition 3 is to rewrite the indicator function of \( (s_i, s_j) \) being an inversion pair as \( \frac{1}{4}(1 + s_i)(1 - s_j) \). Then

\[
f(s) = \sum_{i<j} (1 + s_i)(1 - s_j)/4 = \frac{1}{4} \sum_{i<j} (1 + s_i - s_j - s_isj) = \frac{1}{4} \left( \sum_{i<j} 1 + \sum_{i<j} s_i - \sum_{i<j} s_j - \sum_{i<j} s_isj \right).
\]

In the final expression, the first and fourth sums are independent of \( s \), since for all \( s \), \( \sum_{i<j} 1 \) is \( \frac{n(n-1)}{2} \) and \( \sum_{i<j} s_isj \) is

\[
\left( \frac{a(a-1)}{2} + b(b-1) \right) (1) + (ab) (-1) = \frac{n(n-1)}{2} - 2ab,
\]

so

\[
f(s) = \frac{1}{4} \left( 2ab + \sum_{i<j} s_i - \sum_{i<j} s_j \right).
\]

Since the average value of \( \sum_{i<j} s_i \) over each cyclic orbit equals the average value of \( \sum_{i<j} s_j \) over this orbit\(^2\), the average value of \( f \) over each orbit is \( ab/2 \).

In the particular case \( a = b = 2 \), the six-element set \( S \) decomposes into two orbits, shown in Figure 1. (Here we recode the elements of \( S \) as ordinary bit-strings, representing \(+1\) and \(-1\) by 1 and 0, respectively.) As frequently happens, not all orbits are the same size. But one may also view the orbit of size 2 as a superorbit of size 4, cycling through the same set of elements twice, as discussed below.

A different proof (in keeping with the “equivariant bijection philosophy” discussed in subsection 4.3) associates with each \( s \in S \) the set \( \tilde{s} \subseteq \{1, 2, \ldots, n\} \) consisting of the positions \( 1 \leq i \leq n \) for which \( s_i = -1 \). The collection \( \tilde{S} \) of such sets \( \tilde{s} \) is precisely the set of \( a \)-element subsets of \( \{1, 2, \ldots, n\} \). Then the action of \( \tau \) on \( S \) is isomorphic to the action of \( \tilde{\tau} \) on \( \tilde{S} \), where applying \( \tilde{\tau} \) to \( \tilde{s} \) decrements each element by 1 mod \( n \). Likewise, the inversion statistic \( f \) on \( S \) corresponds to the statistic \( \tilde{f} \) on \( \tilde{S} \), where

\[
\tilde{f}(\tilde{s}) = \left( \sum_{i \in \tilde{s}} i \right) - (1 + 2 + \cdots + a) = \left( \sum_{i \in \tilde{s}} i \right) - \frac{a(a+1)}{2}.
\]

\(^2\)One way to see it is to count how often a given \( s_k \) occurs when we sum the sums \( \sum_{i<j} s_i \) over a given cyclic orbit. It is easy to see that \( s_k \) occurs 0 times in one such sum, 1 times in another; 2 times in another, etc., for a total of \( 0 + 1 + \cdots + (n-1) \) times; but the same can be said of the sum \( \sum_{i<j} s_j \).
Although the orbits of this action can have different sizes, each must be of size \( d \) where \( d \mid n \). So we can repeat such an orbit \( n/d \) times to form a **superorbit** of length \( n \), which has the same average of any statistic as the original orbit. Now each of the \( a \) members of the set \( \tilde{s} \) takes on each value in \( \{1, 2, \ldots, n\} \) over the \( \tilde{\tau} \)-superorbit of \( \tilde{s} \), so that

\[
\sum_{i=0}^{n-1} \tilde{f}(\tilde{\tau}^i \tilde{s}) = a(1 + 2 + \cdots + n) - n \frac{a(a + 1)}{2} = an(n + 1) - \frac{an(a + 1)}{2}
\]

and

\[
\frac{1}{n} \sum_{i=0}^{n-1} \tilde{f}(\tilde{\tau}^i \tilde{s}) = \frac{a(n + 1)}{2} - \frac{a(a + 1)}{2} = \frac{a(n - a)}{2}.
\]

It follows that \((\tilde{S}, \tilde{\tau}, \tilde{f})\), along with \((S, \tau, f)\), is \( c \)-mesic with \( c = a(n - a)/2 = ab/2 \).

A third way to prove Proposition 3 is to derive it from our Theorem 18; see Remark 19.

### 2.4 Linear actions on vector spaces

Let \( V \) be a (not necessarily finite-dimensional) vector space over a field \( K \) of characteristic zero, and define \( f(v) = v \) (that is, our “statistic” is just the identity function). Let \( T : V \to V \) be a linear map such that \( T^n = I \) (the identity map on \( V \)) for some fixed \( n \geq 1 \). Say \( v \) is **invariant** under \( T \) if \( Tv = v \), and **0-mesic** under \( T \) if \( (v + Tv + \cdots + T^{n-1}v)/n = 0 \). Every \( v \in V \) can be written uniquely as the sum of an invariant vector \( \overline{v} \) and a 0-mesic vector \( \hat{v} \). (One suggestive way of paraphrasing the above is: Every element of the kernel of \( I - T^n = (I - T)(I + T + T^2 + \cdots + T^{n-1}) \) can be written uniquely as the sum of an element of the kernel of \( I - T \) and an element of the kernel of \( I + T + T^2 + \cdots + T^{n-1} \).) For, one can check that \( v = \overline{v} + \hat{v} \) is such a decomposition, with \( \overline{v} = (v + Tv + \cdots + T^{n-1}v)/n \) and \( \hat{v} = v - \overline{v} \), and no other such decomposition is possible because that would yield a nonzero vector that is both invariant and 0-mesic, which does not exist. In representation-theoretic terms, we are applying symmetrization to \( v \) to extract from it the invariant component \( \overline{v} \) associated with the trivial representation of the cyclic group, and the homomesic (0-mesic) component \( \hat{v} \) consists of everything else.

This picture relates more directly to our earlier definition if we use the dual space \( V^* \) of linear functionals on \( V \) as the set of statistics on \( V \). As a concrete example, let \( V =
Let \( \mathbb{R}^n \) and let \( T \) be the cyclic shift of coordinates sending \((x_1, x_2, \ldots, x_n)\) to \((x_n, x_1, \ldots, x_{n-1})\). The \( T \)-invariant functionals form a 1-dimensional subspace of \( V^* \) spanned by the functional \( (x_1, x_2, \ldots, x_n) \mapsto x_1 + x_2 + \ldots + x_n \), while the 0-mesic functionals form an \((n - 1)\)-dimensional subspace of \( V^* \) spanned by the \( n-1 \) functionals \((x_1, x_2, \ldots, x_n) \mapsto x_{i} - x_{i+1}(\text{for } 1 \leq i \leq n-1)\).

Also, we can consider the ring \( \mathbb{R}[x_1, \ldots, x_n] \) of polynomial functions \( p(x_1, x_2, \ldots, x_n) \) on \( \mathbb{R}^n \); this ring, viewed as a vector space over \( \mathbb{R} \), can be written as the direct sum of the subspace of polynomials that are invariant under the action of \( T \) and the subspace of polynomials that are 0-mesic under the action of \( T \).

### 2.5 A circle action

Let \( S \) be the set of (real-valued) functions \( f(t) \) satisfying the differential equation \( f''(t) + f(t) = 0 \), that is, the set of functions of the form \( f(t) = A \sin(t - \phi) \), where \( A \) is the amplitude and \( \phi \) is the initial phase. Then we have \( f'' = -f, f''' = -f' \), \( f'''' = f \), etc., so that every polynomial function of \( f, f', f'', f''', \ldots \), can be written as a polynomial in \( f \) and \( f' \).

Evolving \( f \) in time is tantamount to shifting the phase \( \phi \).

Given an element \( p(x, y) \) of the ring \( \mathbb{R}[x, y] \), we will say \( p \) is invariant under time-evolution (or, more compactly, that \( p \) is an invariant) if \( \frac{d}{dt} p(f(t), f'(t)) = 0 \) for all \( f \) in \( S \), and c-mesic if \( \frac{1}{2\pi} \int_0^{2\pi} p(f(t), f'(t)) dt = c \) for all \( f \) in \( S \). For example, \( x^2 + y^2 \) is invariant and \( x \) and \( y \) are 0-mesic; one can think of the first quantity as the total energy of a harmonic oscillator and the second and third as the mean displacement and mean velocity.

We can give a basis for \( \mathbb{R}[x, y] \), viewed as a vector space \( V \) over \( \mathbb{R} \), consisting of the nonnegative powers of \( x^2 + y^2 \) (which jointly span the subspace of \( V \) consisting of all polynomials that are invariant under time-evolution), along with the functions \( x, y, x^2 - y^2, x^3, x^2 y, xy^2, y^3 \), etc. (which jointly span the 0-mesic subspace of \( V \)).

**Proposition 4.** Let \( V_n \) be the \((n+1)\)-dimensional vector subspace of \( \mathbb{R}[x, y] \) spanned by the monomials \( x^a y^b \) with \( a + b = n \). When \( n \) is odd, all of \( V_n \) is 0-mesic. When \( n \) is even, \( V_n \) can be written as the direct sum of an \( n \)-dimensional subspace of 0-mesic functions and a 1-dimensional subspace of functions that are invariant under time-evolution.

**Proof.** Define \( \int_{S^1} p(x, y) = \frac{1}{2\pi} \int_0^{2\pi} p(\cos t, \sin t) dt \), where \( S^1 \) is the unit circle in \( \mathbb{R}^2 \). Consider the monomial \( x^a y^b \) with \( a + b = n \). If \( a \) (resp. \( b \)) is odd, the involution \((x, y) \mapsto (-x, y)\) (resp. \((x, -y)\)) shows that \( \int_{S^1} x^a y^b = 0 \) (using merely that sin is odd and cos is even). If \( a \) and \( b \) are both even, then \( \int_{S^1} x^a y^b \) is some positive number \( c_{a,b} \). Now let \( a, b \) vary subject to \( a + b = n \). If \( n \) is odd, then \( x^a y^b \) is 0-mesic for all \( a, b \) with \( a + b = n \) (since at least one of \( a, b \) is odd), so all of \( V_n \) is 0-mesic. If \( n \) is even, then for \( a, b \) even, \((1/c_{a,b})x^a y^b - (1/c_{a,0})x^n y^0 \) is 0-mesic, and these functions span an \((n/2)\)-dimensional space; adding in the 0-mesic functions \( x^a y^b \) with \( a, b \) odd (and \( a + b = n \)), we get an \( n \)-dimensional space of 0-mesics.

Finally, we must verify that the \( n \)-dimensional space of 0-mesics linearly complements the 1-dimensional space of invariants spanned by \( (x^2 + y^2)^{n/2} \). First we note that (as in subsection 2.4) every function that is both invariant (under time-evolution) and homomesic must be constant; for, any polynomial function \( p(\cdot, \cdot) \) such that the value of \( p(A \cos t, B \sin t) \) is independent of \( t \) (invariance) and independent of \( A \) and \( B \) (homomesy) must be constant.

It follows that the only function in \( V_n \) that is both invariant and 0-mesic is the constant
function 0. Hence the subspace of $V_n$ spanned by 0-mesies and the subspace of $V_n$ spanned by invariants are linearly disjoint. Complementarity then follows from a dimension-count.

Here, as in the preceding section, we get a clean complementarity between invariance and homomesy. That is, every element in $\mathbb{R}[x,y]$ can be written uniquely as the sum of an invariant element and a 0-mesic element.

### 2.6 5-cycles

Let $U$ be the set of all $(x,y)$ in $\mathbb{R}^2$ with $x, y, x + 1, y + 1,$ and $x + y + 1$ all nonzero. The map $\tau : U \to U$ sending $(x,y)$ to $(y,(y + 1)/x)$ has order 5. We can recursively define a sequence $(x_1, x_2, \ldots)$ by $x_1 := x,$ $x_2 := y$ and the (Lyness) recurrence $x_{i-1}x_{i+1} = x_i + 1$, so that $\tau(x_{i-1}, x_i) = (x_i, x_{i+1})$. This sequence turns out to have period 5, thereby giving rise to the Lyness 5-cycle

$$x \rightsquigarrow y \rightsquigarrow (y + 1)/x \rightsquigarrow (x + y + 1)/xy \rightsquigarrow (x + 1)/y \rightsquigarrow x.$$ 

This is associated with the $A_2$ cluster algebra, e.g., by way of four-rowed frieze patterns. (One accessible article on frieze patterns is [Pro08], although it lacks references to many relevant articles written in the past decade.) Let $f((x,y)) = \log |h(x)|$ where $h(z) = z^{-1} + z^{-2}$ is well-defined and nonzero throughout $U$.

**Proposition 5.** The function $f$ is 0-mesic under the action of $\tau$ on $U$.

**Proof.** (Andy Hone) Using the fact that $x_{i-1}x_{i+1} = x_i + 1$ with all subscripts interpreted mod 5 (this is just the Lyness recurrence), we can write the product $h(x_1)h(x_2)h(x_3)h(x_4)h(x_5)$ as $\prod (x_i + 1)/x_i^2 = \prod x_{i+1}x_{i-1}/x_i^2$, and the numerator and denominator factors all cancel, showing that the product is 1.

Applying the map $z \mapsto z/(1 + z)$ to the Lyness 5-cycle we obtain the “Bloch 5-cycle”

$$x \rightsquigarrow y \rightsquigarrow (1-x)/(1-xy) \rightsquigarrow 1-xy \rightsquigarrow (1-y)/(1-xy) \rightsquigarrow x$$

satisfying the recurrence $x_{i-1} + x_{i+1} = x_{i-1}x_{i+1} + 1$. For example, the Lyness 5-cycle

$$\left( \begin{array}{c} 1, 3, 4, 5, 2 \\ 3/3, 3/3, 3/3, 3/3, 3/3 \end{array} \right)$$

maps to the Bloch 5-cycle

$$\left( \begin{array}{c} 1, 3, 4, 5, 2 \\ 2, 4, 5, 8, 5 \end{array} \right).$$

If we let $U'$ be the set of all $(x,y)$ in $\mathbb{R}^2$ with $x, y \not\in \{0,1\}$ and $xy \not= 1$, then the map that sends $(x,y)$ to $(y, (1-x)/(1-xy))$ is an order-5 map from $U'$ to itself, and the Bloch-Wigner function on $C \setminus \{0,1\}$ (a variant of the dilogarithm function; see [W14]) is 0-mesic under this action.

### 2.7 Sandpile dynamics

Let $G$ be a finite directed graph with vertex set $V$. For $v \in V$ let $\text{outdeg}(v)$ be the number of directed edges emanating from $v$, and for $v, w \in V$ let $\text{deg}(v,w)$ be the number of directed edges from $v$ to $w$ (which we will permit to be larger than 1, even when $v = w$). Define the **combinatorial Laplacian** of $G$ as the matrix $\Delta$ (with rows and columns indexed by the
vertices of \( V \) whose \( v, v \)th entry is \( \text{outdeg}(v) - \text{deg}(v, v) \) and whose \( v, w \)th entry for \( v \neq w \) is \( -\text{deg}(v, w) \). Specify a vertex \( t \) with the property that for all \( v \in V \) there is a forward path from \( v \) to \( t \), called the global sink; let \( V^{-} = V \setminus \{ t \} \), and let \( \Delta' \) (the reduced Laplacian) be the matrix \( \Delta \) with the row and column associated with \( t \) removed. By the Matrix-Tree theorem, \( \Delta' \) is nonsingular. A sandpile configuration on \( G \) (with sink at \( t \)) is a function \( \sigma \) from \( V^{-} \) to the nonnegative integers. (For more background on sandpiles, see Holroyd, Levine, Mézéras, Peres, Propp, & Wilson [HLMPPW08].) We say \( \sigma \) is stable if \( \sigma(v) < \text{outdeg}(v) \) for all \( v \in V^{-} \). For any sandpile configuration \( \sigma \), Dhar’s least-action principle for sandpile dynamics (see Levine & Propp [LP10]) tells us that the set of nonnegative-integer-valued functions \( u \) on \( V^{-} \) such that \( \sigma - \Delta'u \) is stable has a unique minimal element \( \phi = \phi(\sigma) \) in the natural (pointwise) ordering; we call \( \phi \) the firing vector for \( \sigma \) and we call \( \sigma - \Delta'\phi \) the stabilization of \( \sigma \), denoted by \( \sigma^\circ \). If we choose a source vertex \( s \in V^{-} \), then we can define an action on sandpile configurations via \( \tau(\sigma) = (\sigma + 1_s)^\circ \), where \( 1_s \) denotes the function that takes the value 1 at \( v \) and 0 elsewhere. Say that \( \sigma \) is recurrent (relative to \( s \)) if \( \tau^m(\sigma) = \sigma \) for some \( m > 0 \). (This notion of recurrence is slightly weaker than that of [HLMPPW08]; they are equivalent when every vertex is reachable by a path from \( s \).) Then \( \tau \) restricts to an invertible map from the set of recurrent sandpile configurations to itself. Let \( f(\sigma) = \phi(\sigma + 1_s) \). Since \( \tau(\sigma) = \sigma + 1_s - \Delta'f(\sigma) \) we have \( \tau(\sigma) = \sigma = 1_s - \Delta'f(\sigma) \); if we average this relation over all \( \sigma \) in a particular \( \tau \)-orbit, the left side telescopes, giving \( 0 = 1_s - \Delta'f \), where \( f \) denotes the average of \( f \) over the orbit. Hence:

**Proposition 6.** Under the action of \( \tau \) on recurrent sandpile configurations described above, the function \( f: \sigma \mapsto \phi(\sigma + 1_s) \) is homomesic, and its orbit-average is the function \( f^* \) on \( V^{-} \) such that \( \Delta'f^* = 1_s \) (unique because \( \Delta' \) is nonsingular).

**Example 7.** Figure 2 shows an example of the \( \tau \)-orbits for the case where \( G \) is the bidirected cycle graph with vertices 1, 2, 3, and 4, with a directed edge from \( i \) to \( j \) iff \( i - j = \pm 1 \mod 4 \); here the discrete Laplacian is

\[
\Delta = \begin{pmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2 \\
\end{pmatrix}.
\]

Let the source be \( s = 2 \) and global sink be \( t = 4 \). The sandpile configuration \( \sigma \) is represented by the triple \((\sigma(1), \sigma(2), \sigma(3))\). The four recurrent configurations \( \sigma \) are \((1, 0, 1), (1, 1, 1), (0, 1, 1)\), and \((1, 1, 0)\), and the respective firing vectors \( f(\sigma) \) are \((0, 0, 0), (1, 2, 1), (0, 1, 1)\), and \((1, 1, 0)\). The average value of the firing vector statistic \( f \) is \( f^* = (\frac{1}{2}, 1, \frac{1}{2}) \) on each orbit. Treating \( f^* \) as a column vector and multiplying on the left by \( \Delta' \) gives the column vector \((0, 1, 0) = 1_s:\)

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
1/2 \\
1 \\
1/2 \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
1 \\
0 \\
\end{pmatrix}.
\]

We should mention that in this situation all orbits are of the same cardinality. This is a consequence of the fact that the set of recurrent sandpile configurations can be given the structure of a finite abelian group (the “sandpile group” of \( G \)). For, given any finite abelian
$\tau \rightarrow (1,0,1) \quad \tau \rightarrow (1,1,1) \quad \tau \rightarrow \ldots$

$\downarrow f \quad \downarrow f$

$(0,0,0) \quad (1,2,1)$

$\tau \rightarrow (0,1,1) \quad \tau \rightarrow (1,1,0) \quad \tau \rightarrow \ldots$

$\downarrow f \quad \downarrow f$

$(0,1,1) \quad (1,1,0)$

Figure 2: The two orbits in the action of the sandpile map $\tau$ on recurrent configurations on the cycle graph of size 4, with source at 2 and sink at 4. There are two orbits, each of size 2, and the average of $f$ along each orbit is $(1/2, 1, 1/2)$. The group $G$ and any element $h \in G$, the action of $h$ on $G$ by multiplication has orbits that are precisely the cosets of $G/H$, where $H$ is the subgroup of $G$ generated by $h$, and all these cosets have size $|H|$. Similar instances of homomesy were known for a variant of sandpile dynamics called rotor-router dynamics; see Holroyd-Propp [HP10]. It was such instances of homomesy that led the second author to seek instances of the phenomenon in other, better-studied areas of combinatorics.

2.8 Suter’s action on Young diagrams

In [Su02], Suter described an action of the dihedral group $D_n$ ($n \geq 1$) on a particular subgraph $Y_n$ of the Hasse diagram of Young’s lattice. Let the hull of a Young diagram be the smallest rectangular diagram that contains it, and let $Y_n$ be all Young diagrams whose hulls are contained in the staircase diagram $(n-1, n-2, \ldots, 1)$. Here we will consider only the cyclic action generated by the invertible operation $\rho_n$ defined by Suter as follows: Given a Young diagram $\lambda \in Y_n$ (drawn “French” style as rows of boxes in the first quadrant) we discard the boxes in the bottom row (let us say there are $k$ of them), move all the remaining boxes one step downward and to the right, and insert a column of $n-1-k$ boxes at the left. For example, the action of $\rho_5$ on $Y_5$ produces the following four orbits:

$$
\begin{pmatrix}
\emptyset, & \ \ & \ \ & \ \ 
\end{pmatrix}, \quad 
\begin{pmatrix}
\emptyset, & \ \ & \ \ & \ \ 
\end{pmatrix}, \\
\begin{pmatrix}
\emptyset, & \ \ & \ \ & \ \ 
\end{pmatrix}, \quad \begin{pmatrix}
\emptyset, & \ \ & \ \ & \ \ 
\end{pmatrix}.
$$

Figure 3 shows another example with $n = 6$ and $k = 2$, where boldface black numbers correspond to boxes that get shifted when one passes from $\lambda$ to $\rho_n(\lambda)$. Suter shows that the map $\rho_n$ is an automorphism of the undirected graph $Y_n$, and that $\rho_n^n$ is the identity on $Y_n$. 

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Let $f$ be the statistic on $Y_n$ that sends each Young diagram to the sum of the weights of its constituent boxes, where the box at the lower left has weight $n-1$, its two neighbors have weight $n-2$, and so on. The boxes in Figure 3 have been marked with their weights, so we can see that $f(\lambda) = 5 + 4 + 4 + 3 + 3 + 2 = 21$ while $f(\rho_6(\lambda)) = 5 + 4 + 4 + 3 + 3 + 3 + 2 = 24$.

**Proposition 8.** Under the action of $\rho_n$ on $Y_n$, the function $f$ is $c$-mesic with $c = (n^3 - n)/12$.

For example, the weights corresponding to each orbit above for $n = 5$ are

$$(0, 10, 15, 15, 10), \quad (4, 9, 14, 14, 9), \quad (7, 12, 12, 12, 7), \quad (10),$$

each of which has average $10 = \frac{5^3 - 5}{12}$.

**Proof.** (David Einstein) The proposition follows from a more refined assertion, in which we take positive integers $i, j$ with $i + j = n$ and only look at the sum of weights of the boxes of weight $i$ and the boxes of weight $j$; call this $f_{i,j}$. We claim that $f_{i,j}$ is homomesic with average $ij$. Then since $f = (f_{1,n-1} + f_{2,n-2} + \cdots + f_{n-1,1})/2$, it will follow that $f$ is homomesic with average $\frac{1}{2}(1)(n-1) + (2)(n-2) + \cdots + (n-1)(1)) = (n-1)(n)(n+1)/12 = \frac{n^3 - n}{12}$.

Note that the diagonal slides in the definition of $\rho_n$ do not affect the weight of a cell. It takes $j$ diagonal sliding operations to move a cell of weight $i$ that starts in the first column so that it disappears, and likewise with the roles of $i$ and $j$ reversed. So each cell of weight $i$ or weight $j$ added in the first column contributes $ij/n$ to the average of $f_{i,j}$.

The definition of $\rho_n$ shows that in going from $\lambda$ to $\rho_n(\lambda)$, we gain cells of weights $n-1, n-2, \ldots, k+1$ and lose cells of weights $n-1, n-2, \ldots, n-k$ (where $k$ is the length of the first row of $\lambda$). So we lose a cell of weight $j$ if and only if we don’t gain a cell of weight $i$. Thus, when we perform $\rho_n$ a total of $n$ times, the number of cells of weight $j$ lost is $n$ minus the number of cells of weight $i$ gained. But the number of cells of weight $j$ gained is the number of cells of weight $j$ lost (what comes in is what comes out). This means that if $r$ cells of weight $j$ are added in a complete cycle, then $n-r$ cells of weight $i$ are added, for a total of $n$ cells of weight either $i$ or $j$. Thus we get an average of $n(ij/n) = ij$ for the sum of the weights of these cells across an orbit. \qed

It should be noted that for this and similar examples, our notion of homomesy of cyclic actions can be adapted in a straightforward fashion to the action of other finite groups.

### 2.9 Rectangular Young tableaux

For a fixed Young diagram $\lambda$, let $\text{SSYT}_k(\lambda)$ denote the set of **semistandard Young tableaux** of shape $\lambda$ and ceiling $k$, i.e., fillings of the cells of $\lambda$ with elements of $[k]$ which are
weakly increasing in each row and strictly increasing in each column. (See, e.g., [Sta99, § 7.10] for more information about these objects and their relationship to symmetric functions.) In the particular case where \( \lambda = (n^m) := (n, n, \ldots, n) \) is a rectangular shape with \( m \) parts, all equal to \( n \), the Schützenberger promotion operator \( \mathcal{P} \) satisfies \( \mathcal{P}^k = \text{id} \) [R10, Cor. 5.6]. (Simpler proofs are available for standard Young tableaux; see e.g. [Sta09, Thm. 4.1(a)] and the references therein.)

Now fix any subset \( R \) of the cells of \( (n^m) \) and for \( T \in \text{SSYT}_k(n^m) \) set \( \sigma_R(T) \) to be the sum of the entries of \( T \) whose cells lie in \( R \).

**Theorem 9** (Bloom-Pechenik-Saracino). Let \( k \) be a positive integer and suppose that \( R \subseteq (n^m) \) is symmetric with respect to 180-degree rotation about the center of \( (n^m) \). Then the statistic \( \sigma_R \) is \( c \)-mesic with respect to the action of promotion on \( \text{SSYT}_k(n^m) \), with \( c = |R| \left( \frac{k+1}{2} \right) \).

For example, consider the following promotion orbit within \( \text{SSYT}_5(3^2) \) (where our tableaux are now drawn “English” style, using matrix coordinates):

\[
\begin{array}{cccc}
1 & 1 & 2 & \\
2 & 3 & 4 & \\
\end{array} \quad \mapsto \quad \begin{array}{cccc}
1 & 1 & 3 & \\
2 & 5 & 5 & \\
\end{array} \quad \mapsto \quad \begin{array}{cccc}
1 & 2 & 4 & \\
4 & 5 & 5 & \\
\end{array} \quad \mapsto \quad \begin{array}{cccc}
1 & 3 & 4 & \\
3 & 4 & 5 & \\
\end{array} \quad \mapsto \quad \begin{array}{cccc}
2 & 2 & 3 & \\
3 & 4 & 5 & \\
\end{array} \quad \mapsto \quad \begin{array}{cccc}
3 & 4 & 5 & \\
\end{array}
\]

Then the sum of the values in the upper left and lower right cells (shown in red) across the orbit is \( (5, 6, 6, 7) \), which averages to \( 6 \). Similarly, the sum of the blue entries in the lower left and upper right corners across the orbit is \( (4, 5, 7, 6) \), with the same average.

This result was stated as a conjecture in several talks given by the authors, and recently proved by J. Bloom, O. Pechenik, and D. Saracino [BPS13]. The latter also prove a version of the result for cominuscule posets. For the action of \( K \)-promotion on increasing tableaux of rectangular shapes, they prove an analogous result for two-rowed shapes, and show that it fails in general when \( \lambda \) is a rectangle with more than two rows.

### 3 Promotion and rowmotion in products of two chains

For a finite poset \( P \), we let \( J(P) \) denote the set of order ideals (or down-sets) of \( P \), \( F(P) \) denote the set of (order) filters (or up-sets) of \( P \), and \( \mathcal{A}(P) \) be the set of antichains of \( P \). (For standard definitions and notation about posets and ideals, see Stanley [Sta11].) There is a bijection \( J(P) \leftrightarrow \mathcal{A}(P) \) given by taking the maximal elements of \( I \in J(P) \) or conversely by taking the order ideal generated by an antichain \( A \in \mathcal{A}(P) \). Similarly, there is a bijection \( F(P) \leftrightarrow \mathcal{A}(P) \). Composing these with the complementation bijection between \( J(P) \) and \( F(P) \) leads to an interesting map that has been studied in several contexts [BS74, Fon93, CF95, Pan09, AST11, SW12], namely \( \Phi_A := \mathcal{A}(P) \to J(P) \to F(P) \to \mathcal{A}(P) \) and the companion map \( \Phi_J := J(P) \to F(P) \to \mathcal{A}(P) \to J(P) \), where the subscript indicates whether we consider the map to be operating on antichains or order ideals. We often drop the subscript and just write \( \Phi \) when context makes clear which is meant. Following Striker and Williams [SW12] we call this map rowmotion.
It should be noted that the maps considered by Brouwer, Schrijver, Cameron, Fon-der-Flaass, and Panyushev are the inverses (“time-reversals” given by toggling in the reverse order) of the maps considered by Striker, Williams, Armstrong, Stump, Thomas, and ourselves; we think that the newer convention is more natural, to the extent it is more natural to cycle through the integers mod \( n \) by repeatedly adding 1 than by repeatedly subtracting 1.

Let \([a] \times [b]\) denote the poset that is a product of chains of lengths \(a\) and \(b\). Figure 6 shows an orbit of the action of \(\Phi_J\) starting from the ideal generated by the antichain \(\{(2,1)\}\). Note that the elements of \([4] \times [2]\) here are represented by the squares rather than the points in the picture, with covering relations represented by shared edges. One can also view this as an orbit of \(\Phi_A\) if one just considers the maximal elements in each shaded order ideal.

This section contains our main specific results, namely that the following triples exhibit homomesy:

\[
(J([a] \times [b]), \Phi_J, \#I); \quad (A([a] \times [b]), \Phi_A, \#A); \quad \text{and} \quad (J([a] \times [b]), \partial_J, \#I).
\]

Here \(\partial_J\) is the promotion operator to be defined in the next subsection, and \(\#I\) (resp. \(\#A\)) denotes the statistic on \(J(P)\) (resp. \(A(P)\)) that is the cardinality of the order ideal \(I\) (resp. the antichain \(A\)). All maps operate on the left (e.g., we write \(\partial_JI\), not \(I\partial_J\)).

### 3.1 Background on the toggle group

Several of our examples arise from the toggle group of a finite poset (first explicitly defined in [SW12]; see also [CF95, Sta09, SW12]). We review some basic facts and provide some pointers to relevant literature.

**Definition 10.** Let \(P\) be a poset. Given \(x \in \text{P}\), we define the toggle operation \(\sigma_x : J(P) \to J(P)\) (“toggling at \(x\”) via

\[
\sigma_x(I) = \begin{cases} 
I \triangle \{x\} & \text{if } I \triangle \{x\} \in J(P); \\
I & \text{otherwise},
\end{cases}
\]

where \(A \triangle B\) denotes the symmetric difference \((A \setminus B) \cup (B \setminus A)\).

**Proposition 11** ([CF95]). Let \(P\) be a poset. (a) For every \(x \in \text{P}\), \(\sigma_x\) is an involution, i.e., \(\sigma_x^2 = 1\).

(b) For every \(x, y \in \text{P}\) where neither \(x\) covers \(y\) nor \(y\) covers \(x\), the toggles commute, i.e., \(\sigma_x \sigma_y = \sigma_y \sigma_x\).

**Proposition 12** ([CF95]). Let \(x_1, x_2, \ldots, x_n\) be any linear extension (i.e., any order-preserving listing of the elements) of a poset \(P\) with \(n\) elements. Then the composite map \(\sigma_{x_1} \sigma_{x_2} \cdots \sigma_{x_n}\) coincides with the rowmotion operator \(\Phi_J\).

Although we do not use the following corollary, it provides context for how we view rowmotion on a finite graded poset.

**Corollary 13** ([SW12], Cor. 4.9). Let \(P\) be a graded poset of rank \(r\), and set \(T_k := \prod_{x \text{ has rank } k} \sigma_x\), the product of all the toggles of elements of fixed rank \(k\). (This is well-defined by Proposition 11.) Then the composition \(T_0 T_1 T_2 \cdots T_r\) coincides with \(\Phi_J\), i.e., rowmotion is the same as toggling by ranks from top to bottom.
We focus on the case \( P = [a] \times [b] \), whose elements we write as \((k, \ell)\).

**Definition 14.** In this situation, we call the sets of \((k, \ell)\) with constant \( k + \ell - 2 \) ranks (in accordance with standard poset terminology), and the sets of \((k, \ell)\) with constant \( \ell - k \) files. More specifically, the element \((k, \ell) \in [a] \times [b]\) belongs to \( \text{rank } k + \ell - 2 \), and to \( \text{file } \ell - k \).

To each order ideal \( I \in J([a] \times [b]) \) we associate a lattice path of length \( a + b \) joining the points \((-a, a)\) and \((b, b)\) in the plane, where each step is of type \((i, j) \rightarrow (i + 1, j - 1)\) or of type \((i, j) \rightarrow (i + 1, j - 1)\), as follows. Given \( 1 \leq k \leq a \) and \( 1 \leq \ell \leq b \), represent \((k, \ell) \in [a] \times [b]\) by the square centered at \((\ell - k, \ell + k - 1)\) with vertices \((\ell - k, \ell + k - 2)\), \((\ell - k, \ell + k)\), \((\ell - k - 1, \ell + k - 1)\), and \((\ell - k + 1, \ell + k - 1)\). (Thus, the poset-elements \((k, \ell) = (1, 1), (a, 1), (1, b), \) and \((a, b)\) are respectively the bottom, left, right, and top squares representing elements of \([a] \times [b]\).) Then the squares representing the elements of the order ideal \( I \) form a “Russian-style” Young diagram whose upper border is a path joining some point on the line of slope \(-1\) to some point on the line of slope \(+1\). Adding extra edges of slope \(-1\) at the left and extra edges of slope \(+1\) at the right if necessary, we get a path joining \((-a, a)\) to \((b, b)\). See Figures 4, 5, and 6 for several examples of this correspondence.

**Definition 15.** We can think of this path as the graph of a (real) piecewise-linear function \( h_I : [-a, b] \rightarrow [0, a + b] \); we call this function (or its restriction to \([-a, b] \cap \mathbb{Z}\) the **height function** representation of the ideal \( I \). Equivalently, for every \( k \in [-a, b] \), we have

\[
h_I(k) = |k| + 2\# \text{ (elements of } I \text{ in file } k \).
\]

In particular, \( h_I(-a) = a \) and \( h_I(b) = b \).

To this height function we can in turn associate a word consisting of \( a \) \(-1\)’s and \( b \) \(+1\)’s, whose \( i \)th letter (for \( 1 \leq i \leq a + b \)) is \( h_I(i - a) - h_I(i - a - 1) = \pm 1 \); we call this the **sign-word** associated with the order ideal \( I \).

Note that the sign-word simply lists the slopes of the segments making up the path, and that either the sign-word or the height-function encodes all the information required to determine the order ideal.

**Proposition 16.** Let \( I \in J([a] \times [b]) \) correspond with height function \( h_I : [-a, b] \rightarrow \mathbb{R} \). Then

\[
\sum_{k=-a}^{b} h_I(k) = \frac{a(a + 1)}{2} + \frac{b(b + 1)}{2} + 2\#I.
\]

So to prove that the cardinality of \( I \) is homomesic, it suffices to prove that the function \( h_I(-a) + h_I(-a + 1) + \cdots + h_I(b) \) is homomesic (where our combinatorial dynamical system acts on height functions \( h \) via its action on order ideals \( I \)).

### 3.2 Promotion in products of two chains

In general a ranked poset \( P \) may not have an embedding in \( \mathbb{Z} \times \mathbb{Z} \) that allows files to be defined; when they are, however, then all toggles corresponding to elements within the same file commute by Proposition 11, so their product is a well defined operation on \( J(P) \). This allows one to define an operation on \( J(P) \) by successively toggling all the files from left to right, in analogy to Corollary 13.

From here on, we set \( P = [a] \times [b] \).
Theorem 17 (Striker-Williams [SW12, § 6.1]). Let $x_1, x_2, \ldots, x_n$ be any enumeration of the elements $(k, \ell)$ of the poset $[a] \times [b]$ arranged in order of increasing $\ell - k$. Then the action on $J(P)$ given by $\partial := \sigma_{x_n} \circ \sigma_{x_{n-1}} \circ \cdots \circ \sigma_{x_1}$ viewed as acting on the paths (or the sign-words representing them) is just a leftward cyclic shift.

Striker and Williams call this well-defined composition $\partial$ promotion (since it is related to Schützenberger’s notion of promotion on linear extensions of posets). They show that it is conjugate to rowmotion in the toggle group, obtaining a much simpler bijection to prove Panyushev’s conjecture in Type A, and generalizing an equivariant bijection for $[a] \times [b]$ of Stanley [Sta09, remark after Thm 2.5]. This definition and their results apply more generally to the class they define of rc-posets, whose elements fit neatly into “rows” and “columns” (which we call here “ranks” and “files”). As with $\Phi$, we can think of $\partial$ as operating either on $J(P)$ or $A(P)$, adding subscripts $\partial_J$ or $\partial_A$ if necessary. Since the cyclic left-shift has order $a + b$, so does $\partial$.

Theorem 18. The cardinality statistic is homomesic under the action of promotion $\partial_J$ on $J([a] \times [b])$, with all averages equal to $ab/2$.

Proof. To show that $\#I$ is homomesic, by Proposition 16 it suffices to show that $h_I(k)$ is homomesic for all $-a \leq k \leq b$. Note that here we are thinking of $I$ as varying over $J(P)$, and $h_I(k)$ (for $I$ varying) as being an $\mathbb{Q}$-valued function on $J(P)$.

We can write $h_I(k)$ as the telescoping sum $h_I(-a) + (h_I(-a + 1) - h_I(-a)) + (h_I(-a + 2) - h_I(-a + 1)) + \cdots + (h_I(k) - h_I(k - 1))$; to show that $h_I(k)$ is homomesic for all $k$, it will be enough to show that all the increments $h_I(k) - h_I(k - 1)$ are homomesic. Note that these
The increments are precisely the letters of the sign-word of $I$. Create a square array with $a + b$ rows and $a + b$ columns, where the rows are the sign-words of $I$ and its successive images under the action of $\partial$; each row is just the cyclic left-shift of the row before. Since each row contains $a - 1$’s and $b + 1$’s, the same is true of each column. Thus, for all $k$, the average value of the $k$th letters of the sign-words of $I, \partial I, \partial^2 I, \ldots, \partial^{a+b-1} I$ is $(b - a)/(b + a)$. This shows that the increments are homomesic, as required, which suffices to prove the theorem. 

Our proof actually shows the more refined result that the restricted cardinality functions $\#(I \cap S)$ where $S$ is any file of $[a] \times [b]$ are homomesic with respect to the action of $\partial_J$.

Remark 19. We now have a third proof of Proposition 3. The bijection sending $I \in J([a] \times [b])$ to its sign-word is an isomorphism between promotion acting on order ideals in $[a] \times [b]$ and the leftward cyclic shift acting on the sign-word. Furthermore, the cardinality of any order ideal is mapped to the number of inversions of the sign-word. So the homomesy of Theorem 18 yields the homomesy of Proposition 3.

The next example shows that the cardinality of the antichain $A_I$ associated with the order ideal $I$ is not homomesic under the action of promotion $\partial$.

Example 20. Consider the two promotion orbits of $\partial_A$ shown in Figures 4 and 5. Although the statistic $\# I$ is homomesic, giving an average of 3 in both cases, the statistic $\# A$ averages to $\frac{1}{5} \left[ 0 + 1 + 1 + 1 + 1 \right] = \frac{4}{5}$ in the first orbit and to $\frac{1}{5} \left[ 1 + 2 + 2 + 1 + 2 \right] = \frac{8}{5}$ in the second.
(2+4+6+6+4+2) / 6 = 4

Figure 6: A rowmotion orbit in $J([4] \times [2])$

### 3.3 Rowmotion in products of two chains

Unlike promotion, the rowmotion operator turns out to exhibit homomesy with respect to both the statistic that counts the size of an order ideal and the statistic that counts the size of an antichain.

#### 3.3.1 Rowmotion on order ideals in $J([a] \times [b])$

We can describe rowmotion nicely in terms of the sign-word. We define a block within any word $w \in \{-1, +1\}^n$ to be an occurrence of the factor $-1, +1$ (that is, a $-1$ followed immediately by a $+1$). A gap in the sign-word is a factor which contains no block; in other words, it is a factor of the form $+1, +1, \ldots, +1, -1, -1, \ldots, -1$. This uniquely decomposes any sign word into blocks and gaps.

Now define the block-gap reversal of $w$ to be the word $\tilde{w}$ obtained by decomposing $w$ into contiguous block and gap subwords, then reversing each subword (leaving the subwords in the same relative order). For example, the binary word

$$w = -1, +1, +1, -1, -1, -1, +1, +1$$

is divided into blocks and gaps as

$$\tilde{w} = +1, -1, +1, -1, +1, -1, +1, +1.$$

Reversing each block and gap in place gives

$$\tilde{\tilde{w}} = +1, -1, -1, +1, +1, -1, -1, +1.$$

or dropping the dividers
Lemma 21. Let $I \in J([a] \times [b])$ correspond to the sign-word $w$, and let $\tilde{w}$ be the block-gap reversal of $w$. Then the sign-word of $\Phi_J(I)$ is $\tilde{w}$. In other words, rowmotion on order ideals is equivalent to block-gap reversal on corresponding sign-words.

Note that (in general) the dividers correspond to the red dots in Figure 6, so one can visualize $\Phi_J$ as reversing ($180^\circ$ rotation of) each lattice-path segment that corresponds to a block or a gap in the sign-word. (See animations within talk slides at http://www.math.uconn.edu/~troby/combErg2012kizugawa.pdf.)

Proof. Consider Figure 6, where the elements of the poset are denoted by the squares (not the dots), and the shaded portions indicate the order ideals to which rowmotion is being applied. Note that the sign-word of $I$ indicates the lattice path that traces out the boundary between $I$ and its complement $I^C$. For example, in the second picture, the lattice path follows the pattern $-1,+1,-1,-1,+1,-1$. Clearly the minimal elements of the complement $I^C$ occur exactly in the locations above where we have a $-1,+1$ pair (indicating a down step followed by an up step as we move from left to right along the lattice path). By definition of rowmotion, these squares become the generators of $\Phi_J(I)$. In particular, each block $-1,+1$ will map to its reversal $+1,-1$, so that these minimal elements of $I^C$ are now maximal in $\Phi_J(I)$.

Now let $G$ be any gap occurring between two blocks $B$ and $B'$ corresponding to the minimal elements $(i,j)$ and $(i',j')$ in $I^C$. Then $G$ must consist of $j' - j - 1 \geq 0$ up steps, followed by $i - i' - 1 \geq 0$ down steps (since the two minimal elements are incomparable). Now by definition of rowmotion, $(i,j)$ and $(i',j')$ are two adjacent maximal elements of $\Phi_J(I)$, and so the part of the sign-word of $\Phi_J(I)$ between the corresponding $+1,-1$ segments will have the form $-1,-1,\ldots,-1,+1,+1,\ldots,+1$. Thus, (this is especially clear if one creates a generic diagram like those in Figure 6) the lattice path segment that corresponds to this part consists of $i - i' - 1$ down steps followed by $j' - j - 1$ up steps. Similar arguments handle the cases where the gap occurs at the beginning or end of the sign-word.

It turns out that all we really need to know for purposes of proving homomesy is that the sign-word for $I$ has $-1,+1$ in a pair of adjacent positions if and only if the sign-word for $\Phi_J(I)$ has $+1,-1$ in the same two positions. This can be seen directly for $J([a] \times [b])$ from the description of $\Phi_J$ given at the start of Section 3. (See also Figure 6.) This situation occurs if and only if the antichain $A(\Phi_J(I))$ contains an element in the associated file of $[a] \times [b]$.

Theorem 22. The cardinality statistic is homomesic under the action of rowmotion $\Phi_J$ on $J([a] \times [b])$, with all averages equal to $ab/2$.

Proof. As in the previous section, to prove that $\#I$ is homomesic under rowmotion, it suffices to prove that all the increments $h_I(k) - h_I(k-1)$ are homomesic. There is a positive integer $N$ such that $\Phi^N = \text{id}$ (since $J([a] \times [b])$ is finite\(^3\)). Now, proving that $h_I(k) - h_I(k-1)$ is homomesic is equivalent to showing that for all $k$, the sum of the $k$th letters of the sign-words of $\Phi^0 I, \Phi^1 I, \ldots, \Phi^{N-1} I$ is independent of $I$.

\(^3\)A result of Fon-der-Flaass [Fon93, Theorem 2] states that the size of any $\Phi$-orbit in $[a] \times [b]$ is a divisor of $a + b$ (this also follows from Proposition 25), so that we can take $N = a + b$; but any $N$ will do here.
Create a rectangular array with $N$ rows and $a + b$ columns, where the rows are the sign-words of $I$ and its successive images under the action of $\Phi$. Consider any two consecutive columns of the array, and the width-2 subarray they form. There are just four possible combinations of values in a row of the subarray: $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, and $(-1, -1)$. However, we have just remarked that a row is of type $(-1, +1)$ if and only if the next row is of type $(+1, -1)$ (where we consider the row after the bottom row to be the top row). Hence the number of rows of type $(-1, +1)$ equals the number of rows of type $(+1, -1)$. It follows that any two consecutive column-sums of the full array are equal, since other row types contribute the same value to each column sum. That is, within the original rectangular array, every two consecutive columns have the same column-sum. Hence all columns have the same column-sum. This common value of the column-sum must be $1/(a + b)$ times the grand total of the values of the rectangular array. But since each row contains $a$ $-1$’s and $b$ $+1$’s, each row-sum is $b - a$, so the grand total is $N(b - a)$, and each column-sum is $N(b - a)/(a + b)$. Since this is independent of which rowmotion orbit we are in, we have proved homomesy for letters of the sign-word of $I$ as $I$ varies over $J([a] \times [b])$, and this gives us the desired result about $\#I$, just as in the proof of Theorem 18.

3.3.2 Rowmotion on antichains in $A([a] \times [b])$

In his survey article on promotion and evacuation, Stanley [Sta09, remark after Thm 2.5] gave a concrete equivariant bijection between rowmotion $\Phi_A$ acting on antichains in $A([a] \times [b])$ and cyclic rotation of certain words on \{1, 2\}. Armstrong (private communication) gave a variant description that clarified the correspondence, which he learned from Thomas and which we use in what follows.
Definition 23. Fix $a$, $b$, and $n = a + b$. For every given $k \in [a]$, we call the subset \{(k, \ell) : \ell \in [b]\} of $[a] \times [b]$ the $k$-th row. For every given $\ell \in [b]$, we call the subset \{(k, \ell) : k \in [a]\} of $[a] \times [b]$ the $\ell$-th column. We use the word \textbf{fber} to mean a row or a column. Define the Stanley-Thomas word $w(A)$ of an antichain $A$ in $[a] \times [b]$ to be $w_1 w_2 \cdots w_{a+b} \in \{-1, +1\}^{a+b}$ with

$$w_i := \begin{cases} +1, & \text{if } A \text{ has an element in row } i \text{ (for } i \in [a] \text{) or} \\ -1, & \text{if } A \text{ has NO element in column } i - a \text{ (for } a + 1 \leq i \leq n); \\ \end{cases}$$

Example 24. As illustrated in Figure 7, let $A = \{(1,5),(5,3),(6,2)\}$. By definition, the Stanley-Thomas word $w(A)$ should have +1 in entries 1, 5, and 6 (rows where $A$ appears) and in entries 8 and 11 (columns where $A$ does not appear, with indices shifted by 7 = $a$). Indeed one sees that $w(A) = +1, -1, -1, +1, +1, -1, -1, +1, -1, -1, -1$. Indeed one sees that $w(A) = +1, -1, -1, +1, +1, -1, -1, +1, -1, -1, -1$. Note that applying rowmotion gives $A' = \Phi(A) = \{(2,4),(6,3),(7,1)\}$ with Stanley-Thomas word $w(A') = -1, +1, -1, -1, +1, +1, -1, +1, -1, -1, +1 = C_R w(A)$, the rightward cyclic shift of $w(A)$.

Proposition 25 (Stanley-Thomas). The correspondence $A \leftrightarrow w(A)$ is a bijection from $\mathcal{A}([a] \times [b])$ to binary words $w \in \{-1, +1\}^{a+b}$ with exactly $a$ occurrences of $-1$ and $b$ of $+1$. Furthermore, this bijection is equivariant with respect to the actions of rowmotion $\Phi_A$ and rightward cyclic shift $C_R$.

Note that the classical result that $\Phi_A^{a+b}$ is the identity map follows immediately.

Proof. Let $\mathcal{W}_{a,b}$ denote the set of binary words in $\{-1, +1\}^{a+b}$ with exactly $a$ occurrences of $-1$ and $b$ of $+1$. The map $A \mapsto w(A)$ is clearly well-defined into $\{-1, +1\}^{a+b}$. By definition, the number of occurrences of $+1$ among the first $a$ indices is $\#A$; among the remaining $b$ indices, it is $b - \#A$, giving a total of $b$ occurrences of $+1$ in $w(A)$. Thus, $w(A) \in \mathcal{W}_{a,b}$.

This map has an inverse as follows. Given any word $u \in \mathcal{W}_{a,b}$, let $k$ denote the number of indices $1 \leq i \leq a$ with $u_i = +1$, and let $1 \leq i_1 < i_2 < \cdots < i_k \leq a$ denote those indices. There must be $a - k$ occurrences of $-1$ among these indices, hence, $k$ occurrences of $-1$ among the remaining $b$ entries (since the total must sum to $a$ by definition of $\mathcal{W}_{a,b}$). Let $a + 1 \leq j_1' < j_2' < \cdots < j_k' \leq a + b$ denote those indices corresponding to $-1$, and set $j_{\ell} := j_{\ell}' - a$. Then the corresponding antichain $A$ is given by

$$A = \{(i_1,j_k),(i_2,j_{k-1}),\ldots,(i_k,j_1)\}.$$ 

Note that this is the only pairing of the indices that gives an antichain in $[a] \times [b]$. It follows from the definitions that for this $A$, $w(A) = u$, whence $w$ is a bijection between $\mathcal{A}([a] \times [b])$ and $\mathcal{W}_{a,b}$.

It remains to show that the following diagram commutes:

$$\begin{array}{ccc} 
\mathcal{A}([a] \times [b]) & \xrightarrow{w} & \mathcal{W}_{a,b} \\
\Phi_A \downarrow & & \downarrow C_R \\
\mathcal{A}([a] \times [b]) & \xrightarrow{w} & \mathcal{W}_{a,b} 
\end{array}$$
To that end, let $A$ be any antichain in $\mathcal{A}([a] \times [b])$, and set $A' := \Phi(A)$, $u := w(A)$ and $u' := w(A')$. We want to show that $u' = C_R u$.

Recall our initial definition at the start of this section of $\Phi_A$ as the composition

$$
\Phi_A : \mathcal{A}(P) \to J(P) \to F(P) \to \mathcal{A}(P)
$$

$$
A \mapsto A \mapsto I_A \mapsto \overline{I_A} \mapsto A',
$$

where $A'$ is the minimal elements of the complement of the order ideal $I_A$ generated by $A$. Suppose first that $i \in [a - 1]$. If $u_i = +1$, then there is an antichain element $(i, j)$ in row $i$, which is not the top row. Because $A$ is an antichain, any element of $A$ in row $i + 1$ must lie in a column $j' < j$. (This includes the case when there is no element of $A$ in row $i + 1$.) This means that the complement $\overline{I_A}$ of the corresponding order ideal will have a minimal element in row $i + 1$. (A glance at Figure 7 should make this clear.) Thus, by definition of $\Phi_A$, $A'$ will have an element in row $i + 1$, so $u'_{i+1} = +1$.

On the other hand, if $u_i = -1$, then no element of $A$ lies in row $i$. This means that the rightmost column $j$ of $I_A$ in rows $i$ and $i + 1$ must be the same. (Possibly $j = 0$ if $I_A$ has empty intersection with row $i$.) But then both $(i, j + 1)$ and $(i + 1, j + 1)$ lie in $\overline{I_A}$, so $\overline{I_A}$ has no minimal elements in row $i + 1$. (Visually this is saying that the lattice path defining $I_A$ has two successive down-steps, corresponding to two successive $\text{−}1$’s of the sign-word, in rows $i + 1$ and $i$.) Thus, in $\overline{I_A}$, no minimal element occurs in row $i + 1$, so $u'_{i+1} = -1$.

Similar arguments show that for $j \in [b - 1]$, $u'_{a+j+1} = u_{a+j}$. It remains only to check positions $a$ and $a + b$ in $u$.

If $u_a = 1$, then $I_A$ includes all of column 1. Therefore, $\overline{I_A}$, hence $A'$, has no elements at all in column 1, and $u'_{a+1} = +1$ by definition. On the other hand, if $u_a = -1$, then $I_A$ includes only a proper subset of column 1. This means that $\overline{I_A}$, hence $A'$, must have elements in column 1, since the only elements smaller than an element in column 1 also lie in column 1. Thus, $u'_{a+1} = -1$.

A similar argument shows that $u'_1 = u_{a+b}$, and we have $u' = C_R u$ as required.

\begin{theorem}
The cardinality statistic is homomesic under the action of rowmotion $\Phi_A$ on $\mathcal{A}([a] \times [b])$, with all averages equal to $ab/(a + b)$.
\end{theorem}

**Proof.** It suffices to prove a more refined claim, namely, that if $S$ is any row or column of $[a] \times [b]$, the cardinality of $A \cap S$ is homomesic under the action of rowmotion on $A$. By the previous result, rowmotion corresponds to cyclic shift of the Stanley-Thomas word, and the entries in the Stanley-Thomas word tell us which fibers (rows or columns) contain an element of $A$ and which do not. Specifically, for $1 \leq k \leq a$, if $S$ is the $k$th row, then $A$ intersects $S$ iff the $k$th symbol of the Stanley-Thomas word is a $+1$. Since the Stanley-Thomas word contains $a - 1$’s and $b$ $+1$’s, the multiset orbit of $A$ of size $a + b$ has exactly $b$ elements that are antichains that intersect $S$. That is, the sum of $\#(A \cap S)$ over the multiset orbit of size $a + b$ is exactly $b$, for each of the $a$ rows of $[a] \times [b]$. Summing over all the rows, we see that the sum of $\#A$ over the multiset orbit is $ab$. Hence $\#A$ is homomesic with average $ab/(a + b)$. \qed
4 Summary

First we summarize what we know about the specific case of products of two chains, going beyond what is proved here, citing results that will be proved in follow-up articles. Then we discuss how the case of \([a] \times [b]\) can be conceived of as a small component of a larger research program. Lastly, we offer some thoughts about directions that this research program might take.

4.1 Rowmotion and promotion for order ideals and antichains

A natural way to find homomesies for the action of a map \(\tau\) on some combinatorial set \(S\) is to start with some finite set of not necessarily homomesic functions \(f_1, f_2, \ldots, f_N\) associated with the combinatorial presentation of the set \(S\), and then to inquire which linear combinations of the \(f_i\)’s are homomesic. For example, if \(S\) is the set of order ideals of a poset \(P\), then for each element \(x \in P\) we have an indicator function \(1_x : S \to \{0, 1\}\) such that \(f_x(I)\) is 1 if \(x \in I\) and 0 otherwise. We look in the span of the functions \(f_i\) (call it \(V\)); the functions in \(V\) that satisfy homomesy form a subspace of \(V\) whose intersection with the subspace of invariants functions in \(V\) is trivial.

In the case of rowmotion acting on order ideals of \([a] \times [b]\), we find that the function \(\sum_{x \in F} 1_x\) is homomesic whenever \(F\) is a file of \([a] \times [b]\). Also, \(1_x + 1_y\) is homomesic whenever \(x\) and \(y\) are opposite elements of \([a] \times [b]\) (that is, they are obtained from one another by rotating the poset 180 degrees about its center). These can be shown to generate the subspace of homomesies.

The situation is the same for promotion acting on order ideals of \([a] \times [b]\). That is because of the extremely intimate relationship between rowmotion and promotion, as seen for instance in Theorem 5.4 of [SW12].

In the case of rowmotion acting on antichains of \([a] \times [b]\), the situation is different. Now \(S\) is the set of antichains of a poset \(P\), and for each element \(x \in P\) we have an indicator function \(1_x : S \to \{0, 1\}\) such that \(1_x(A)\) is 1 if \(x \in A\) and 0 otherwise. Although this vector space, like the one considered above, is \(|P|\)-dimensional, there is no way to write the \(|P|\) indicator functions we have just defined as linear combinations of the \(|P|\) indicator functions considered above. Hence there is no reason to expect the subspace of homomesies for antichains to have anything to do with the subspace of homomesies for order ideals.

For rowmotion on antichains, we find that the function \(\sum_{x \in F} 1_x\) is homomesic whenever \(F\) is a fiber of \([a] \times [b]\). Also, \(1_x - 1_y\) is 0-mesic whenever \(x\) and \(y\) are opposite elements of \([a] \times [b]\). These can be shown to generate the subspace of homomesies.

For promotion on antichains, the situation is not so clear. One thing we do know is that the homomesic subspace under the action of promotion is not the same as the homomesic subspace under the action of rowmotion (Theorem 5.4 of [SW12] cannot be applied here). In particular, the total cardinality statistic is not homomesic in this case. However, other statistics are homomesic. A natural open problem is to settle this fourth case.

More broadly one can ask the same sorts of questions when \([a] \times [b]\) is replaced by other rc-posets in the terminology of [SW12] (or, more properly speaking, other posets with a specified rc-embedding). Preliminary work by the authors and others suggests that typically the subspace of homomesies is substantial.
It should be stressed that the choice of an ambient space of statistics plays a key role
in determining what one finds. The action of rowmotion on order ideals is conjugate to the
action of promotion on antichains, but in choosing between the order ideals picture and the
antichains picture one is choosing between two different spaces of statistics (one generated
by the indicator functions arising from order ideals and the other generated by the indicator
functions arising from antichains). Since these are two different spaces, their homomesic
subspaces can be (and are) different. As a more trivial example, note that if one considers
the (huge) vector space spanned by the indicator functions $1_s : S \rightarrow \{0, 1\}$ ($s \in S$) such
that $1_s(s')$ is 1 if $s = s'$ and 0 otherwise, then the space of homomesies is large but not very
interesting, as it reflects only the orbit-structure of the action, in a very simple way.

4.2 Cyclic sieving

We have observed informally that the sorts of combinatorial objects that exhibit the cyclic
sieving phenomenon also tend to exhibit the “homomesy phenomenon” (by which we mean,
the abundance of homomesies). It is natural to ask whether the connection goes both
ways. We think the answer is No. Specifically, we can construct examples of (conjectural)
homomesy in which the order of the cyclic group generated by $\tau$ is much larger than the
size of $S$ (e.g., $|S| = 377$ while the order of $\tau$ exceeds 3 million). This is very unlike typical
instances of the CSP, for which we have actions of small cyclic groups on large combinatorial
sets.

4.3 Equivariant bijections

Given the role that equivariant bijections play in the proofs of homomesy results, one might
come to the view that the bijections are what is truly fundamental, while the homomesies
are mere epiphenomena. We have some sympathy for this point of view. Those leaning in
this direction should view homomesies as empirical indicators of the existence of (known or
unknown) equivariant bijections whose unearthing renders the homomesies explicable.

However, it should be borne in mind that some homomesy results do not follow from the
existence of a single equivariant bijection, but from the existence of many of them; that is,
sometimes a function is shown to be homomesic by breaking it down as a linear combination
of components that are separately homomesic, where different components require different
equivariant bijections. It’s also worth noting that not all equivariant bijections used to prove
homomesy are with objects that are being cyclically rotated, e.g., Lemma 21. Above all, the
homomesy point of view brings to the fore the notion that homomesies form a vector space.

4.4 Complementarity

In section 2 we saw several cases in which the ambient vector space can be written as the
direct sum of the subspace of homomesic functions and the subspace of invariant functions.
These were the cases in which the map $\tau$ was in some sense linear. This kind of sharp
complementarity between the notions of homomesy and invariance is not seen (or at least
not seen in a naively recognizable form) for more interesting, nonlinear maps like rowmotion
and promotion. However, it is possible that a more sophisticated view – possibly one that
recognizes that the set of invariant functions has much richer closure properties than the set of homomesic functions – would enable us to retrieve something like this sharp complementarity in many nonlinear settings.

4.5 Promising avenues

We have already mentioned that situations in which cyclic sieving has been observed strike us as good places to dig in search of homomesies. One example is the CSP proved by Brendon Rhoades [R10].

As another example, we mention the study of rowmotion on the product of three chains. What are the homomesies for the action of rowmotion on order ideals or antichains in \([a] \times [b] \times [c]\)? Preliminary study indicates that non-trivial homomesies may exist.

Toggles as discussed in subsection 3.1 can be viewed in the more general context of flipping in polytopes. This point of view was first proposed (in a special case) in [KB95] and is developed more fully in [EP13]. Products of toggles in this geometrical setting seem like a likely source of interesting homomesies.

It would be extremely interesting if homomesies showed up in the discrete dynamical systems associated with cluster algebras. The example of subsection 2.6 suggests that cluster algebras of type A (associated with frieze patterns) might be a natural place to look.

References


