Math 329 (Roby) **Representation Theory HW4** Due 4 May 2007, noon

1. Show that for all $X \in M_n(\mathbb{C})$

$$\lim_{m\to\infty} \left[I + \frac{X}{m} \right]^m = e^X .$$

2. (a) Show that the following matrices form a basis for the real Lie algebra $\mathfrak{su}(2)$:

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

   (b) Show that $\mathfrak{su}(2)$ and $\mathfrak{sl}(2;\mathbb{R})$ are not isomorophic Lie algebras, even though $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{R})_{\mathbb{C}} \cong \mathfrak{sl}(2;\mathbb{C})$.

3. Let $X \in \mathfrak{gl}(n;\mathbb{C})$ be a diagonalizable matrix. Show that ad_X is a diagonalizable operator on $\mathfrak{gl}(n;\mathbb{C})$.

4. Show that the subalgebra of $\mathfrak{sl}(3;\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   is isomorphic to $\mathfrak{sl}(2;\mathbb{C})$. This lets us view $\mathfrak{sl}(3,\mathbb{C})$ as an $\mathfrak{sl}(2,\mathbb{C})$-module, with action given by $x \cdot y = [x, y]$ for $x \in \mathfrak{sl}(2,\mathbb{C})$ and $y \in \mathfrak{sl}(3,\mathbb{C})$. Find the irreducible decomposition of this module.

5. Let $V \cong \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_2(\mathbb{C})$. Show that for $a \geq b$

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V = \mathrm{Sym}^{a+b} V \oplus \mathrm{Sym}^{a+b-2} V \oplus \cdots \oplus \mathrm{Sym}^{a-b} V .$$