Math 329 (Roby) **Representation Theory HW3** **Due 17 April 2007**

1. Find all $f \in \Lambda^n$ for which $\omega f = 2f$.

2. (a) Expand the power series $\displaystyle\prod_{i\geq 1}(1 + x_i + x_i^2)$ in terms of the elementary symmetric functions. (Hint available on request.)

   (b) For what real numbers a is the symmetric formal power series

   $$F(x) = \prod_{i\geq 1}(1 + x_i + ax_i^2)$$

   e-positive, i.e., a nonnegative (infinite) linear combination of the e_λ's?

3. Find the number $y(n)$ of pairs (λ, μ) such that $\lambda \vdash n$ and μ covers λ in Young's lattice $\mathbb{Y}$. Express your answer in terms of $p(k)$, the number of partitions of k, for certain values of k. Try to give a direct bijection, avoiding generating functions, recurrence relations, induction, etc.

4. Let Ω^n denote the subspace of Λ^n consisting of all $f \in \Lambda^n$ satisfying

   $$f(x_1, -x_1, x_3, x_4 \ldots) = f(x_3, x_4 \ldots)\,.$$

   For instance, $m_1 = x_1 + x_2 + \cdots \in \Omega^1$. Find a "simple" basis for Ω^n. Express the dimension of Ω^n in terms of the number of partitions of n with a suitable restriction.

5. Let $f \in \Lambda^n$, and for any $g \in \Lambda^n$ define $g_k \in \Lambda^{nk}$ by

   $$g_k(x_1, x_2, \ldots) = g(x_1^k, x_2^k, \ldots)\,.$$

   Show that

   $$\omega f_k = (-1)^{n(k-1)}(\omega f)_k\,.$$

6. Let $\ell(\lambda) \leq m$ and $\lambda_1 \leq n$. Define

   $$\tilde{\lambda} = (n - \lambda_m, n - \lambda_{n-1}, \ldots, n - \lambda_1)\,.$$

   (a) Give a simple (algebraic or combinatorial) proof that

   $$(x_1 x_2 \cdots x_m)^n s_\lambda(x_1^{-1}, \ldots, x_m^{-1}) = s_{\tilde{\lambda}}(x_1, \ldots, x_m).$$

   (b) Show that

   $$\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i + y_j) = \sum_{\lambda} s_\lambda(x) s_{\tilde{\lambda}}(y)$$

   summed over all partitions λ with $\ell(\lambda) \leq m$ and $\lambda_1 \leq n$.