Math 3250/Roby **Practice Midterm Solutions** October 2008

This is a closed book, closed note exam. Calculators are not permitted. Please do not discuss this exam with anyone other than the proctor during the exam. **SHOW ALL YOUR WORK!** Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

1. Go over all the old homework problems, with particularly attention to anything that gave you trouble.

2. Go over all the problems (with solutions) in Bona listed for quiz preparation.

3. Go over all the entries of *The Twelvefold Way* and makes sure you understand how each entry was derived.

   We did this in class...

4. An encyclopedia has 24 volumes. How many selections of 5 volumes are there with no two consecutive volumes chosen? (The order of selection is immaterial.)

   This is like Bona, Example 3.19 (p. 45). The answer is $\binom{24-5+1}{5} = \binom{20}{5} = 15,504$.

5. How many ways are there to write the digits from 0 to 9 such that each number except for the leftmost is within one of some number to the left of it?

   Think of writing the numbers from right to left: $b_0, b_1, \ldots, , b_9$*. Then we claim that* b_9 *must be 0 or 9. For 8 must lie left of 9, and 7 must lie left of 8, etc. until we hit the leftmost digit* b_0*; similarly,* $1, 2, 3, \ldots, b_0$ *must lie to left of 0. Therefore, each* $i \in [8]$ *lies to the left of either 0 or of 9, which means only 0 or 9 could be the rightmost digit.*

   Now we write an arbitrary such sequence from RIGHT to LEFT. We have two choices for b_9*, and the argument above shows inductively that at each step,* b_i *= highest or lowest choice of the remaining numbers (*$\neq b_{i+1}, \ldots, b_9$*). So there are* $2^9 = 512$ *possible choices.*

6. Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x)$ and use this to find a closed form expression for r_n itself.

   Carefully taking into account the initial terms, the recursive formula for r_n *yields that* $R(x) - xR(x) - 6x^2R(x) = 3 + x$*. Therefore,*

$$R(x) = \frac{3+x}{1-x-6x^2} = \frac{1}{1+2x} + \frac{2}{1-3x},$$

   where the last equality uses the method of partial fractions. Now expanding these two geometric series yields the explicit formula $r_n = (-2)^n + 2(3)^n$ *for* $n \geq 0$.

7. Let r be any irrational number. Prove that there exists a positive integer n such that the distance of nr from the closest integer is less than 10^{-10}.

   Represent the set of all positive real numbers on a circle of radius 1, so two numbers that differ by an integer are the same point of the circle. (This is like taking $\mathbb{R}/\mathbb{Z}$ or ignoring the integer part of the number.) Now divide the circumference of the circle into 10^{10} equal arcs, and consider the first $10^{10}+1$ terms of the sequence $r, 2r, 3r, \ldots,$. By the Pigeonhole Principle, there must be some arc that contains two different multiples of r, say mr and nr, with $m < n$. Then $nr - mr = (n-m)r$ must be less than 10^{-10} on the circle, so $(n-m)r$ lies within 10^{-10} of an integer on the real line itself.

8. Let $p(n) =$ number of integer partitions of n; prove that $\sum_{i=1}^{n} p(i) < p(2n)$ for all $n \in \mathbb{Z}^+$.

   See Bona Ex.5.14 and solution.

9. Instead of studying for their combinatorics midterm, four students play Pac-Man each hour and compare their scores. How many hours will it take before they can be sure to get the same relative ordering of scores? (All kinds of ties are possible, e.g., W scores more than X and Y, who tie and whose score is greater than Z.)

   This problem is isomorphic to Bona Ex. 5.15, whose solution is in the text.

10. a) Prove that the ordinary generating function for the sequence $c_n = \binom{2n}{n}$ is $(1-4x)^{-\frac{1}{2}}$.

    See Bona Ex. 4.26 (and solution).

    b) Prove that

$$\sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i} = 4^n .$$

    Interpret the LHS of this as the coefficient of x^n in the product of the OGF of (a) with itself, whose closed form expression is $(1-4x)^{-\frac{1}{2}} \cdot (1-4x)^{-\frac{1}{2}} = \frac{1}{(1-4x)}$.

    c) (Extra credit) Can you give a combinatorial proof?

    This is harder than it looks...