1.7 Linear Independence

A homogeneous system such as

\[
\begin{bmatrix}
1 & 2 & -3 \\
3 & 5 & 9 \\
5 & 9 & 3 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

can be viewed as a vector equation

\[
x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

The vector equation has the trivial solution \((x_1 = 0, x_2 = 0, x_3 = 0)\), but is this the only solution?

Definition

A set of vectors \(\{v_1, v_2, \ldots, v_p\}\) in \(\mathbb{R}^n\) is said to be **linearly independent** if the vector equation

\[
x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0
\]

has only the trivial solution. The set \(\{v_1, v_2, \ldots, v_p\}\) is said to be **linearly dependent** if there exists weights \(c_1, \ldots, c_p\), not all 0, such that

\[
c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0.
\]

**linear dependence relation**

(when weights are not all zero)
EXAMPLE Let $v_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\langle v_1, v_2, v_3 \rangle$ is linearly independent.

b. If possible, find a linear dependence relation among $v_1, v_2, v_3$.

Solution: (a)

$$
\begin{align*}
\begin{bmatrix} x_1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{align*}
$$

Augmented matrix:

$$
\begin{bmatrix}
1 & 2 & -3 & 0 \\
3 & 5 & 9 & 0 \\
5 & 9 & 3 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & -3 & 0 \\
0 & -1 & 18 & 0 \\
0 & -1 & 18 & 0 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & -3 & 0 \\
0 & -1 & 18 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

$x_3$ is a free variable $\Rightarrow$ there are nontrivial solutions.

$\langle v_1, v_2, v_3 \rangle$ is a linearly dependent set

(b) Reduced echelon form:

$$
\begin{align*}
\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\Rightarrow
\begin{align*}
x_1 &= x_2 = 0 \\
x_3 &
\end{align*}
\end{align*}
$$

Let $x_3 = ____$ (any nonzero number). Then $x_1 = ____$ and $x_2 = ____$.

$$
\begin{align*}
\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
$$

or

$$
____v_1 + ____v_2 + ____v_3 = 0
$$

(one possible linear dependence relation)
Linear Independence of Matrix Columns

A linear dependence relation such as

\[
\begin{bmatrix}
-3 & 3 \\
3 & 5 \\
5 & 9 \\
\end{bmatrix}
+ 18
\begin{bmatrix}
2 \\
5 \\
9 \\
\end{bmatrix}
+ 1
\begin{bmatrix}
-3 \\
9 \\
3 \\
\end{bmatrix}
= 0
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

can be written as the matrix equation:

\[
\begin{bmatrix}
1 & 2 & -3 \\
3 & 5 & 9 \\
5 & 9 & 3 \\
\end{bmatrix}
\begin{bmatrix}
-33 \\
18 \\
1 \\
\end{bmatrix}
= 0
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

Each linear dependence relation among the columns of \( A \) corresponds to a nontrivial solution to \( A \mathbf{x} = \mathbf{0} \).

The columns of matrix \( A \) are linearly independent if and only if the equation \( A \mathbf{x} = \mathbf{0} \) has only the trivial solution.

Special Cases

Sometimes we can determine linear independence of a set with minimal effort.

1. **A Set of One Vector**

Consider the set containing one nonzero vector: \( \{ \mathbf{v}_1 \} \)

The only solution to \( x_1 \mathbf{v}_1 = \mathbf{0} \) is \( x_1 = \underline{\phantom{0}} \).

So \( \{ \mathbf{v}_1 \} \) is linearly independent when \( \mathbf{v}_1 \neq \mathbf{0} \).
2. A Set of Two Vectors

EXAMPLE Let

\[ \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]

a. Determine if \( \{ \mathbf{u}_1, \mathbf{u}_2 \} \) is a linearly dependent set or a linearly independent set.

b. Determine if \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that \( \mathbf{u}_2 = \ldots \mathbf{u}_1 \). Therefore

\[ \ldots \mathbf{u}_1 + \ldots \mathbf{u}_2 = 0 \]

This means that \( \{ \mathbf{u}_1, \mathbf{u}_2 \} \) is a linearly ______________ set.

(b) Suppose

\[ c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}. \]

Then \( \mathbf{v}_1 = \ldots \mathbf{v}_2 \) if \( c \neq 0 \). But this is impossible since \( \mathbf{v}_1 \) is \ldots a multiple of \( \mathbf{v}_2 \) which means \( c = \ldots \).

Similarly, \( \mathbf{v}_2 = \ldots \mathbf{v}_1 \) if \( d \neq 0 \). But this is impossible since \( \mathbf{v}_2 \) is not a multiple of \( \mathbf{v}_1 \) and so \( d = 0 \). This means that \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is a linearly ______________ set.

A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.
3. **A Set Containing the 0 Vector**

**Theorem 9**

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\mathbf{v}_1 = \underline{\text{____}}$. Then

$$\underline{\text{____}} \mathbf{v}_1 + \underline{\text{____}} \mathbf{v}_2 + \cdots + \underline{\text{____}} \mathbf{v}_p = \mathbf{0}$$

which shows that $S$ is linearly ________________.

4. **A Set Containing Too Many Vectors**

**Theorem 8**

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Outline of Proof:

$$A = \left[ \begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{array} \right] \text{ is } n \times p$$

Suppose $p > n$.

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has more variables than equations

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has nontrivial solutions

$\Rightarrow$ columns of $A$ are linearly dependent

**EXAMPLE**  With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. \[ \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} , \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\} \]

b. Columns of \[ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix} \]
Characterization of Linearly Dependent Sets

**EXAMPLE** Consider the set of vectors $\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^3$ in the following diagram. Is the set linearly dependent? Explain

[Diagram of vectors $v_1, v_2, v_3, v_4$ in $\mathbb{R}^3$]

**Theorem 7**

An indexed set $S = \{v_1, v_2, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent, and $v_1 \neq 0$, then some vector $v_j$ ($j \geq 2$) is a linear combination of the preceding vectors $v_1, \ldots, v_{j-1}$.