4.5 The Dimension of a Vector Space

**THEOREM 9**

If a vector space $V$ has a basis $\beta = \{b_1, \ldots, b_n\}$, then any set in $V$ containing more than $n$ vectors must be linearly dependent.

**Proof:** Suppose $\{u_1, \ldots, u_p\}$ is a set of vectors in $V$ where $p > n$. Then the coordinate vectors $\{[u_1]_\beta, \ldots, [u_p]_\beta\}$ are in $\mathbb{R}^n$. Since $p > n$, $\{[u_1]_\beta, \ldots, [u_p]_\beta\}$ are linearly dependent and therefore $\{u_1, \ldots, u_p\}$ are linearly dependent. ■

**THEOREM 10**

If a vector space $V$ has a basis of $n$ vectors, then every basis of $V$ must consist of $n$ vectors.

**Proof:** Suppose $\beta_1$ is a basis for $V$ consisting of exactly $n$ vectors. Now suppose $\beta_2$ is any other basis for $V$. By the definition of a basis, we know that $\beta_1$ and $\beta_2$ are both linearly independent sets.

By Theorem 9, if $\beta_1$ has more vectors than $\beta_2$, then ______ is a linearly dependent set (which cannot be the case).

Again by Theorem 9, if $\beta_2$ has more vectors than $\beta_1$, then ______ is a linearly dependent set (which cannot be the case).

Therefore $\beta_2$ has exactly $n$ vectors also. ■

**DEFINITION**

If $V$ is spanned by a finite set, then $V$ is said to be **finite-dimensional**, and the **dimension** of $V$, written as $\dim V$, is the number of vectors in a basis for $V$. The dimension of the zero vector space $\{0\}$ is defined to be 0. If $V$ is not spanned by a finite set, then $V$ is said to be **infinite-dimensional**.

**EXAMPLE:** The standard basis for $P_3$ is $\{\}$.

In general, $\dim P_n = n + 1$.

**EXAMPLE:** The standard basis for $\mathbb{R}^n$ is $\{e_1, \ldots, e_n\}$ where $e_1, \ldots, e_n$ are the columns of $I_n$. So, for example, $\dim \mathbb{R}^3 = 3$. 


EXAMPLE: Find a basis and the dimension of the subspace 

\[ W = \left\{ \begin{bmatrix} a + b + 2c \\ 2a + 2b + 4c + d \\ b + c + d \\ 3a + 3c + d \end{bmatrix} : a, b, c, d \text{ are real} \right\}. \]

Solution: Since 

\[
\begin{bmatrix}
  a + b + 2c \\
  2a + 2b + 4c + d \\
  b + c + d \\
  3a + 3c + d
\end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},
\]

\[ W = \text{span} \{v_1, v_2, v_3, v_4\} \text{ where} \]

\[
\begin{align*}
v_1 &= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \\
v_2 &= \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \\
v_3 &= \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \\
v_4 &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.
\end{align*}
\]

- Note that \(v_3\) is a linear combination of \(v_1\) and \(v_2\), so by the Spanning Set Theorem, we may discard \(v_3\).

- \(v_4\) is not a linear combination of \(v_1\) and \(v_2\). So \(\{v_1, v_2, v_4\}\) is a basis for \(W\).

- Also, \(\text{dim } W =\).
EXAMPLE: Dimensions of subspaces of $R^3$

0-dimensional subspace contains only the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

1-dimensional subspaces. Span $\{v\}$ where $v \neq 0$ is in $R^3$.

These subspaces are ____________ through the origin.

2-dimensional subspaces. Span $\{u, v\}$ where $u$ and $v$ are in $R^3$ and are not multiples of each other.

These subspaces are ____________ through the origin.

3-dimensional subspaces. Span $\{u, v, w\}$ where $u, v, w$ are linearly independent vectors in $R^3$. This subspace is $R^3$ itself because the columns of $A = \begin{bmatrix} u & v & w \end{bmatrix}$ span $R^3$ according to the IMT.

THEOREM 11

Let $H$ be a subspace of a finite-dimensional vector space $V$. Any linearly independent set in $H$ can be expanded, if necessary, to a basis for $H$. Also, $H$ is finite-dimensional and 
\[ \dim H \leq \dim V. \]

EXAMPLE: Let $H = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then $H$ is a subspace of $R^3$ and $\dim H < \dim R^3$.

We could expand the spanning set $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ to form a basis for $R^3$. 
THEOREM 12  THE BASIS THEOREM
Let $V$ be a $p$-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly $p$ vectors in $V$ is automatically a basis for $V$. Any set of exactly $p$ vectors that spans $V$ is automatically a basis for $V$.

EXAMPLE: Show that $\{t, 1-t, 1+t-t^2\}$ is a basis for $P_2$.

Solution: Let $v_1 = t, v_2 = 1-t, v_3 = 1+t-t^2$ and $\beta = \{1,t,t^2\}$.

Corresponding coordinate vectors

\[
[v_1]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [v_2]_\beta = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, [v_3]_\beta = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}
\]

$[v_2]_\beta$ is not a multiple of $[v_1]_\beta$

$[v_3]_\beta$ is not a linear combination of $[v_1]_\beta$ and $[v_2]_\beta$

$\Rightarrow \{[v_1]_\beta, [v_2]_\beta, [v_3]_\beta\}$ is linearly independent and therefore $\{v_1, v_2, v_3\}$ is also linearly independent.

Since $\dim P_2 = 3$, $\{v_1, v_2, v_3\}$ is a basis for $P_2$ according to The Basis Theorem.

Dimensions of Col $A$ and Nul $A$

Recall our techniques to find basis sets for column spaces and null spaces.

EXAMPLE: Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find $\dim Col A$ and $\dim Nul A$.

Solution

\[
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]

So $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \\ 8 \end{bmatrix}\right\}$ is a basis for $\text{Col} A$ and $\dim \text{Col} A = 2$. 
Now solve $Ax = 0$ by row-reducing the corresponding augmented matrix. Then we arrive at

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 0 \\
2 & 4 & 7 & 8 & 0 \\
\end{bmatrix} \sim \cdots \sim
\begin{bmatrix}
1 & 2 & 0 & 4 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
$$

$x_1 = -2x_2 - 4x_4$

$x_3 = 0$

and

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = x_2 \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
\end{bmatrix} + x_4 \begin{bmatrix}
-4 \\
0 \\
0 \\
1 \\
\end{bmatrix}
$$

So $\left\{ \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-4 \\
0 \\
0 \\
1 \\
\end{bmatrix} \right\}$ is a basis for $\text{Nul } A$ and

$\dim \text{Nul } A = 2.$

Note

$$\dim \text{Col } A = \text{number of pivot columns of } A$$

$$\dim \text{Nul } A = \text{number of free variables of } A.$$