

**SHOW ALL YOUR WORK!** Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! No calculators are to be used, but you may bring one 4"  $\times$  6" notecard to class with any notes you like.

- For each of the following statements, indicate whether the statement is true or false. If true, give a short explanation of why (full proofs unnecessary). If false, provide a counterexample. Try to "salvage" the false statements by providing a similar (non-trivial) statement that would be true.
  - Let  $V$  be a finite-dimensional vector space and  $T \in \mathcal{L}(V)$ . Then any intersection of  $T$ -invariant subspaces of  $V$  is also  $T$ -invariant.
  - If  $\lambda$  is a nonzero eigenvalue of  $ST$ , then  $\lambda$  is a nonzero eigenvalue of  $TS$ .
  - Let  $V$  be a finite-dimensional vector space and  $T \in \mathcal{L}(V)$ . Then  $T$  has an eigenvalue.
  - Every linear operator on a complex finite-dimensional vector space can be diagonalized.
  - A linear operator on an  $n$ -dimensional vector space can be diagonalized iff it has  $n$  distinct eigenvalues.
- Prove or give a counterexample: if  $T \in \mathcal{L}(V)$  and  $c \in \mathbf{F}$ , then  $\det(cT) = c^{\dim V} \det T$ .
- Prove or give a counterexample: if  $T \in \mathcal{L}(V)$  and  $c \in \mathbf{F}$ , then  $\text{trace}(cT) = c \text{trace } T$ .
- Suppose  $N \in \mathcal{L}(V)$  is nilpotent. Prove directly (from the definition) that 0 is the only eigenvalue of  $N$ .
- Suppose  $T \in \mathcal{L}(V, W)$ . Prove that
  - $T$  is injective if and only if  $T^*$  is surjective;
  - $T$  is surjective if and only if  $T^*$  is injective.
- Suppose  $V$  is a complex inner-product space. Prove that every normal operator on  $V$  has a square root. (An operator  $S \in \mathcal{L}(V)$  is called a *square root* of  $T \in \mathcal{L}(V)$  if  $S^2 = T$ .)
- Suppose  $P \in \mathcal{L}(V)$  and  $P^2 = P$ . Prove that  $V = \text{null } P \oplus \text{range } P$ .
- Prove or disprove: the identity operator on  $\mathbf{R}^2$  has infinitely many self-adjoint square roots.
- Go back over your old homework, quizzes and midterm to review and make sure you understand any problem on which you lost points. Compare them with the solutions handed out or done in class.