Matrix Diagonals

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A discussion of certain ideas related to the diagonals and diagonal structure of a square matrix, with connections to the polytope of doubly stochastic matrices, its simplex faces, ray-nonsingularity, determinantal regions, digraphs, and ... .
$S_n$: all permutations of $\{1, 2, \ldots, n\}$

\[
\begin{pmatrix}
1 & 2 & \cdots & n \\
i_1 & i_2 & \cdots & i_n \\
\end{pmatrix}
\]

corresponding to positions $(1, i_1), (2, i_2), \ldots, (n, i_n)$ of a matrix of order $n$. 
$S_n$: all permutations of $\{1, 2, \ldots, n\}$

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\end{pmatrix}
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corresponding to positions $(1, i_1), (2, i_2), \ldots, (n, i_n)$ of a matrix of order $n$.

$S_n \leftrightarrow \mathcal{P}_n$, the set of permutation matrices of order $n$, that is, $(0, 1)$-matrices of order $n$ with exactly one 1 in each row and column,
$S_n$: all permutations of $\{1, 2, \ldots, n\}$

$\left( \begin{array}{cccc} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{array} \right)$

corresponding to positions $(1, i_1), (2, i_2), \ldots, (n, i_n)$ of a matrix of order $n$.

$S_n \leftrightarrow P_n$, the set of permutation matrices of order $n$, that is, $(0, 1)$-matrices of order $n$ with exactly one 1 in each row and column,
Nonzero diagonals
Nonzero diagonals

In a matrix $A = [a_{ij}]$ of order $n$, a nonzero diagonal consists of the $n$ positions $(k, i_k)$ of a permutation (matrix), occupied by nonzero entries (or the $n$ nonzero entries $a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n}$ themselves).
Nonzero diagonals

In a matrix $A = [a_{ij}]$ of order $n$, a **nonzero diagonal** consists of the $n$ positions $(k, i_k)$ of a permutation (matrix), occupied by nonzero entries (or the $n$ nonzero entries $a_{1i_1}, a_{2i_2}, \ldots, a_{ni_n}$ themselves).

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 5 & 3 & 1 & 4
\end{pmatrix} \rightarrow
\begin{bmatrix}
0 & 2 & 1 & 5 & 0 \\
4 & 2 & 0 & 0 & 1 \\
5 & 3 & 5 & 0 & 6 \\
2 & 0 & 3 & 1 & 2 \\
0 & 7 & 2 & 8 & 0
\end{bmatrix}.
$$
Frobenius-König Theorem
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F-K Theorem: The matrix $A$ of order $n$ has a nonzero diagonal iff it has no $O_{pq}$ with $p + q = n + 1$. 
Frobenius-König Theorem

F-K Theorem: The matrix $A$ of order $n$ has a nonzero diagonal iff it has no $O_{pq}$ with $p + q = n + 1$.

Equivalently, the nonzero entries of $A$ cannot be covered by fewer than $n$ rows and columns: after permutations of rows and columns

$$A = \begin{bmatrix}
(p \times n - q) & O_{pq} \\
(n - p \times n - q) & (n - p \times q)
\end{bmatrix},$$

where $(n - p) + (n - q) = 2n - (p + q) = 2n - (n + 1) = n - 1$. 
Some ideas in terms of $S_n$ or $P_n$
Some ideas in terms of $S_n$ or $\mathcal{P}_n$

determinant: $\sum_{\sigma \in S_n} \pm a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)}$. 
Some ideas in terms of $S_n$ or $P_n$

determinant: $\sum_{\sigma \in S_n} \pm a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

permanent: $\sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

(In both cases one need only sum over the nonzero diagonals)
Some ideas in terms of $S_n$ or $P_n$

**Determinant:** $\sum_{\sigma \in S_n} \pm a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

**Permanent:** $\sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

(In both cases one need only sum over the nonzero diagonals)

**Total support:** every nonzero entry belongs to a nonzero diagonal; e.g. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Some ideas in terms of $S_n$ or $P_n$

determinant: $\sum_{\sigma \in S_n} \pm a_1\sigma(1)a_2\sigma(2) \cdots a_n\sigma(n)$. 
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\begin{pmatrix}
1 & 1 & 0 \\
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fully indecomposable: total support and deletion of a row and column always leaves a matrix with a nonzero diagonal; e.g. 
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\begin{pmatrix}
1 & 1 & 0 \\
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\end{pmatrix}
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determinant: $\sum_{\sigma \in S_n} \pm a_1\sigma(1)a_2\sigma(2) \cdots a_n\sigma(n)$.

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\begin{bmatrix}
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\]
Let $A$ be a matrix of order $n$ whose nonzero entries are labeled with different symbols $V = \{a, b, c, \ldots\}$. The diagonal hypergraph $D(A)$ of $A$ has vertex set $V$ and edges (of size $n$) corresponding to the nonzero diagonals of $A$. 
Diagonal Hypergraph

Let $A$ be a matrix of order $n$ whose nonzero entries are labeled with different symbols $V = \{a, b, c, \ldots\}$. The **diagonal hypergraph** $D(A)$ of $A$ has vertex set $V$ and edges (of size $n$) corresponding to the nonzero diagonals of $A$.

**Example:**

\[
A = \begin{bmatrix}
a & b & 0 \\
0 & c & d \\
e & f & g \\
\end{bmatrix}
\]

\{\{a, c, g\}, \{a, d, f\}, \{b, d, e\}\}.
Diagonal Hypergraph

Let $A$ be a matrix of order $n$ whose nonzero entries are labeled with different symbols $V = \{a, b, c, \ldots\}$. The **diagonal hypergraph** $\mathcal{D}(A)$ of $A$ has vertex set $V$ and edges (of size $n$) corresponding to the nonzero diagonals of $A$.

**Example:** 

$$
A = \begin{bmatrix}
  a & b & 0 \\
  0 & c & d \\
  e & f & g \\
\end{bmatrix}
\quad \{\{a, c, g\}, \{a, d, f\}, \{b, d, e\}\}.
$$

Let $B$, a matrix of order $n$, have the same number of nonzero entries as $A$. $A$ and $B$ have **isomorphic diagonal hypergraphs** provided the nonzero entries of $B$ can be labeled with $V$ so that $\mathcal{D}(A) = \mathcal{D}(B)$. The resulting bijection between the nonzero entries of $A$ and those of $B$ is a **diagonal preserver**.
Example
Example:

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$$A = \begin{bmatrix}
0 & a & b & 0 & 0 \\
c & d & e & 0 & 0 \\
f & 0 & g & h & i \\
0 & 0 & j & 0 & k \\
0 & 0 & 0 & l & m \\
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
0 & a & b & 0 & 0 \\
c & d & e & 0 & 0 \\
f & 0 & g & j & 0 \\
0 & 0 & h & 0 & l \\
0 & 0 & i & k & m \\
\end{bmatrix}$$

have the same nonzero diagonals:

$$\{a, c, g, k, l\}, \{a, c, j, h, m\}, \{a, c, j, i, l\}, \{a, f, e, k, l\},$$

so we have a **diagonal preserver** between $A$ and $B$. 
Example:

A = \begin{bmatrix}
0 & a & b & 0 & 0 \\
c & d & e & 0 & 0 \\
f & 0 & g & h & i \\
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0 & 0 & 0 & l & m \\
\end{bmatrix}\quad \text{and} \quad B = \begin{bmatrix}
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c & d & e & 0 & 0 \\
f & 0 & g & j & 0 \\
0 & 0 & h & 0 & l \\
0 & 0 & i & k & m \\
\end{bmatrix}

have the same nonzero diagonals:

\{a, c, g, k, l\}, \{a, c, j, h, m\}, \{a, c, j, i, l\}, \{a, f, e, k, l\},

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Example:

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A = \begin{bmatrix}
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& c & d & e & 0 & 0 \\
& & f & 0 & g & h & i \\
& & & 0 & 0 & j & 0 \\
& & & & 0 & 0 & l & m
\end{bmatrix}
\]

and 

\[
B = \begin{bmatrix}
0 & a & b & 0 & 0 \\
& c & d & e & 0 & 0 \\
& & f & 0 & g & j & 0 \\
& & & 0 & 0 & h & 0 & l \\
& & & & 0 & 0 & i & k & m
\end{bmatrix}
\]

have the same nonzero diagonals:

\[
\{a, c, g, k, l\}, \{a, c, j, h, m\}, \{a, c, j, i, l\}, \{a, f, e, k, l\},
\]

so we have a diagonal preserver between \(A\) and \(B\). Row and column permutations and transposition give diagonal preservers.
Example again
Example again

Example:

\[
A = \begin{bmatrix}
0 & a & b & 0 & 0 \\
c & d & e & 0 & 0 \\
f & 0 & g & h & i \\
0 & 0 & j & 0 & k \\
0 & 0 & 0 & l & m
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
0 & a & b & 0 & 0 \\
c & d & e & 0 & 0 \\
f & 0 & g & j & 0 \\
0 & 0 & h & 0 & l \\
0 & 0 & i & k & m
\end{bmatrix}
\]
Example again

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\[
A = \begin{bmatrix}
0 & a & b & 0 & 0 \\
c & d & e & 0 & 0 \\
f & 0 & g & h & i \\
0 & 0 & j & 0 & k \\
0 & 0 & 0 & l & m
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & a & b & 0 & 0 \\
c & d & e & 0 & 0 \\
f & 0 & g & j & 0 \\
0 & 0 & h & 0 & l \\
0 & 0 & i & k & m
\end{bmatrix}
\]

have the same nonzero diagonals:

\[
\{a, c, g, k, l\}, \{a, c, j, h, m\}, \{a, c, j, i, l\}, \{a, f, e, k, l\},
\]

so we have a diagonal preserver between \(A\) and \(B\) but it is not a row or column permutation or transposition, or composition of such.
Partial transposition
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The preceding example is an example of a partial transposition:
Partial transposition

The preceding example is an example of a **partial transposition**:

\[
\begin{bmatrix}
** & \alpha & \beta \\
\gamma & A_1 & O \\
\delta & O & A_2 \\
\end{bmatrix} \rightarrow
\begin{bmatrix}
** & \alpha & \delta^T \\
\gamma & A_1 & O \\
\beta^T & O & A_2^T \\
\end{bmatrix}
\]
Partial transposition

The preceding example is an example of a **partial transposition**:

$$
\begin{bmatrix}
** & \alpha & \beta \\
\gamma & A_1 & O \\
\delta & O & A_2 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
** & \alpha & \delta^T \\
\gamma & A_1 & O \\
\beta^T & O & A_2^T \\
\end{bmatrix}
$$

**Note:**

1. Transposition can be regarded as a special case of partial transposition (the matrix $A_1$ is empty).
2. Like row and column permutations, a partial transposition preserves nonzero diagonals.
3. A partial transposition requires a $p$ by $q$ zero submatrix and a complementary $q$ by $p$ zero submatrix where $p + q = n - 1$. 
A Conjecture
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RAB and J. Ross Conjecture 1981: Every diagonal preserver between matrices $A$ and $B$ of order $n$ can be accomplished by a sequence of partial transpositions, followed by row and column permutations.
A Conjecture

RAB and J. Ross Conjecture 1981: Every diagonal preserver between matrices $A$ and $B$ of order $n$ can be accomplished by a sequence of partial transpositions, followed by row and column permutations.

This was proved under the assumption that the labels of $n$ nonzero positions of $A$ are the labels of $n$ nonzero positions in a row or column of $B$. Recall the example:

$$A = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & h & i \\ 0 & 0 & j & 0 & k \\ 0 & 0 & 0 & l & m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & b & 0 & 0 \\ c & d & e & 0 & 0 \\ f & 0 & g & j & 0 \\ 0 & 0 & h & 0 & l \\ 0 & 0 & i & k & m \end{bmatrix}$$
Counterexample to the conjecture
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\[
A = \begin{bmatrix}
a & b & c & 0 & 0 & 0 \\
d & e & f & 0 & 0 & 0 \\
0 & 0 & k & l & m & 0 \\
0 & 0 & r & s & t & 0 \\
w & 0 & 0 & 0 & u & v \\
z & 0 & 0 & 0 & x & y \\
\end{bmatrix}
\]

and \[
A^* = \begin{bmatrix}
a & b & c & 0 & 0 & 0 \\
d & e & f & 0 & 0 & 0 \\
0 & 0 & u & v & w & 0 \\
0 & 0 & x & y & z & 0 \\
m & 0 & 0 & 0 & k & l \\
t & 0 & 0 & 0 & r & s \\
\end{bmatrix}
\]

have the same set of 16 diagonals.
Counterexample to the conjecture

\[ A = \begin{bmatrix}
  a & b & c & 0 & 0 & 0 \\
  d & e & f & 0 & 0 & 0 \\
  0 & 0 & k & l & m & 0 \\
  0 & 0 & r & s & t & 0 \\
  w & 0 & 0 & 0 & u & v \\
  z & 0 & 0 & 0 & x & y \\
\end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix}
  a & b & c & 0 & 0 & 0 \\
  d & e & f & 0 & 0 & 0 \\
  0 & 0 & u & v & w & 0 \\
  0 & 0 & x & y & z & 0 \\
  m & 0 & 0 & 0 & k & l \\
  t & 0 & 0 & 0 & r & s \\
\end{bmatrix} \]

have the same set of 16 diagonals.

But no partial transposition is possible since there do not exist complementary \( p \) by \( q \) and \( q \) by \( p \) zero submatrices with \( p + q = 5 \).
Counterexample to the conjecture

\[
A = \begin{bmatrix}
 a & b & c & 0 & 0 & 0 \\
 d & e & f & 0 & 0 & 0 \\
 0 & 0 & k & l & m & 0 \\
 0 & 0 & r & s & t & 0 \\
 w & 0 & 0 & 0 & u & v \\
 z & 0 & 0 & 0 & x & y \\
\end{bmatrix}
\quad \text{and} \quad
A^* = \begin{bmatrix}
 a & b & c & 0 & 0 & 0 \\
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 0 & 0 & u & v & w & 0 \\
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 m & 0 & 0 & 0 & k & l \\
 t & 0 & 0 & 0 & r & s \\
\end{bmatrix}
\]

have the same set of 16 diagonals.

But no partial transposition is possible since there do not exist complementary \( p \) by \( q \) and \( q \) by \( p \) zero submatrices with \( p + q = 5 \).

So the conjecture is false.
Example again
Example again

\[
\begin{bmatrix}
  a & b & c & 0 & 0 & 0 \\
  d & e & f & 0 & 0 & 0 \\
  0 & 0 & k & l & m & 0 \\
  0 & 0 & r & s & t & 0 \\
  w & 0 & 0 & 0 & u & v \\
  z & 0 & 0 & 0 & x & y \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  a & b & c & 0 & 0 & 0 \\
  d & e & f & 0 & 0 & 0 \\
  0 & 0 & u & v & w & 0 \\
  0 & 0 & x & y & z & 0 \\
  m & 0 & 0 & 0 & k & l \\
  t & 0 & 0 & 0 & r & s \\
\end{bmatrix}
\]

(a diagonal preserver)
Bi-transposition
Bi-transposition

The preceding example is an example of what we call a bi-transposition:

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f
\end{bmatrix} \leftarrow A_1, \ p \text{ by } p + 1
\]

\[
\begin{bmatrix}
  k & l & m \\
  r & s & t
\end{bmatrix} \leftarrow A_2, \ q \text{ by } q + 1
\]

\[
\begin{bmatrix}
  w & u & v \\
  z & x & y
\end{bmatrix} \leftarrow A_3, \ r \text{ by } r + 1
\]
Characterization of diagonal preservers
RAB, Loebl, Pangrác 2006: Let $A$ and $B$ be fully indecomposable matrices of order $n$. Suppose there is a bijection $\phi$ between the nonzero positions of $A$ and those of $B$ that gives a bijection between the nonzero diagonals of $A$ and those of $B$. Then $\phi$ results from a sequence of **partial transpositions** and **bi-transpositions** which when applied to $A$ give $PBQ$ or $PBTQ$ for some permutation matrices $P$ and $Q$. 
**Doubly Stochastic Matrices**

**doubly stochastic matrix**: a square nonnegative matrix with all row and column sums equal to 1.
Doubly Stochastic Matrices

doubly stochastic matrix: a square nonnegative matrix with all row and column sums equal to 1. Equivalently, by the F-K theorem, a convex combination of permutation matrices; e.g.

\[
\begin{bmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix}
= \frac{1}{4}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

\[+
\frac{1}{4}
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
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\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
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0 & 1 & 0 \\
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\frac{1}{2} & \frac{1}{2} & 0 \\
\end{bmatrix}
= \frac{1}{4}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}.
\]

The doubly stochastic matrices of order \( n \) form a convex polytope \( \Omega_n \) of dimension \((n - 1)^2\).
Matrices with total support are geometric objects. **Faces** of $\Omega_n$ correspond to the $(0, 1)$-matrices $A = [a_{ij}]$ of order $n$ with total support:

$$A \leftrightarrow \mathcal{F}(A) = \{X = [x_{ij}], X \in \Omega_n, x_{ij} \leq a_{ij}\}.$$
Doubly stochastic polytope $\Omega_n$

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**Extreme points** are the permutation matrices of order $n$ with $P$ and $Q$ determining an edge (1-dim. face) iff, with respect to $P$, $Q$ has a unique cycle of length $\geq 2$, that is $P^{-1}Q$ has a unique cycle of length $\geq 2$. 
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$\text{per} A = \text{number of extreme points of the face } F(A)$. 
Doubly stochastic polytope $\Omega_n$

Matrices with total support are geometric objects. Faces of $\Omega_n$ correspond to the $(0, 1)$-matrices $A = [a_{ij}]$ of order $n$ with total support:

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**Extreme points** are the permutation matrices of order $n$ with $P$ and $Q$ determining an edge (1-dim. face) iff, with respect to $P$, $Q$ has a unique cycle of length $\geq 2$, that is $P^{-1}Q$ has a unique cycle of length $\geq 2$.

$\text{per}A = \text{number of extreme points of the face } F(A)$.

If $A$ is fully indecomposable, then $\dim F = \#(A) - 2n + 1$. 
Example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$, fully indecomposable, $\text{per} \ A = 9$. 
Example

\[ A = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{bmatrix}, \text{ fully indecomposable, } \text{per} A = 9. \]

\( \mathcal{F}(A) \) is an 8-dimensional simplex in \( \mathbb{R}^{25} \).
Determinantal regions

Let $A$ be a complex matrix of order $n$ whose entries are either $0$ or of the form $e^{i\theta}$. The determinantal region of $A$ is

$$R_{\text{det}}(A) = \{\det X \circ A : X \text{ is a positive matrix}\}.$$ 

The matrix is ray-nonsingular iff $0 \notin R_{\text{det}}(A)$. 
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The matrix is **ray-nonsingular** iff $0 \notin R_{\text{det}}(A)$.

For example,

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -i & -1 \\ -1 & 1 & -i \\ -1 & i & -1 \end{bmatrix}$$

The signed diagonal products of the latter matrix are $1, -1, i$. Since $0$ is on the boundary of the region they determine, it is ray-nonsingular.
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Isolated Sets of Diagonals

(MacDonald, Olesky, Tsatsomeros, van den Driessche 1997) The set $\mathcal{D}(A)$ of nonzero diagonals of $A$ is an isolated set of diagonals of $A$ provided that
Isolated Sets of Diagonals

(MacDonald, Olesky, Tsatsomeros, van den Driessche 1997) The set $\mathcal{D}(A)$ of nonzero diagonals of $A$ is an isolated set of diagonals of $A$ provided that

- Every nonzero diagonal in $\mathcal{D}(A)$ contains an entry which is not in any of the other diagonals of $\mathcal{D}(A)$. 
Example

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]
Example

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

Then \(|D(A)| = 4\) and \(D(A)\) is an isolated set of diagonals of \(A\).
Example

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \rightarrow \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}. \]

Then \(|\mathcal{D}(A)| = 4\) and \(\mathcal{D}(A)\) is an isolated set of diagonals of \(A\).

Let \(A\) be a complex matrix. Any positive linear combination of the signed diagonal products corresponding to an isolated set of diagonals of \(A\) is in the closure of \(R_{\det}(A)\). (Don’t need closure if the isolated set of diagonals is \(\mathcal{D}(A)\).)
Example continued

$$\text{det} \left( \begin{bmatrix} y_1 & y_2 & 1 \\ y_3 & y_4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \right) =$$
Example continued

\[
\begin{vmatrix}
\begin{bmatrix}
y_1 & y_2 & 1 \\
y_3 & y_4 & 1 \\
1 & 1 & 1
\end{bmatrix}
\end{vmatrix}
\circ
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & 0
\end{bmatrix}
\]

\[
y_1(-a_{11}a_{23}a_{32}) + y_2(a_{12}a_{23}a_{31}) + y_3(a_{13}a_{21}a_{32}) + y_4(-a_{13}a_{22}a_{31})
\]

is in \( R_{\text{det}}(A) \).
Digraph of a matrix

If $A = [a_{ij}]$ is a matrix of order $n$, then the digraph $\Gamma(A)$ has vertices $1, 2, \ldots, n$ and edges $i \rightarrow j$ provided that $a_{ij} \neq 0$. 
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For example, the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ has digraph
Characterization I of $\mathcal{D}(A)$ isolated

(J. Shao, L.-Z Ren, and Q. Wu 2006/07): WLOG assume that $I_n \leq A$. Then $\mathcal{D}(A)$ is an isolated set of diagonals iff the following two conditions are satisfied:
Characterization I of $D(A)$ isolated

(J. Shao, L.-Z Ren, and Q. Wu 2006/07): WLOG assume that $I_n \leq A$. Then $D(A)$ is an isolated set of diagonals iff the following two conditions are satisfied:

1. In the digraph $\Gamma(A)$ of $A$ there is a vertex that belongs to all directed cycles of length $\geq 1$.

2. $\Gamma(A)$ is free-cyclic meaning that each cycle contains an edge that is not an edge of any other directed cycle.
Characterization II of $\mathcal{D}(A)$ isolated

(J. Shao, L.-Z Ren, and Q.Wu): WLOG assume that $I_n \leq A$. Then $\mathcal{D}(A)$ is an isolated set of diagonals iff the following two conditions are satisfied:
Characterization II of \( \mathcal{D}(A) \) isolated

(J. Shao, L.-Z Ren, and Q.Wu): WLOG assume that \( I_n \leq A \). Then \( \mathcal{D}(A) \) is an isolated set of diagonals iff the following two conditions are satisfied:

- In the digraph \( \Gamma(A) \) of \( A \) there is a vertex that belongs to all directed cycles of length \( > 1 \).
- \( \Gamma(A) \) does not contain a subdigraph consisting of a cycle

\[
1 \to 2 \to 3 \to \cdots \to k \to 1
\]

along with “path-chords”

\[
1 \to \cdots \to p \text{ and } q \to \cdots \to r
\]

where \( 3 \leq p < q < r \leq k \).
Let $A$ be a fully indecomposable matrix of order $n \geq 2$. Then the following are equivalent:
Characterization III of $\mathcal{D}(A)$ isolated

Let $A$ be a fully indecomposable matrix of order $n \geq 2$. Then the following are equivalent:

- $\mathcal{D}(A)$ is an isolated set of diagonals.
- $A$ has property $N$, and cannot be contracted to $J_3$.
- There exists an integer $p$ with $1 \leq p \leq n - 2$ and permutation matrices $P$ and $Q$ such that
  
  \[
  PAQ = \begin{bmatrix}
  A_3 & A_1 \\
  A_2 & O
  \end{bmatrix}
  \]

  where $A_3$ is $n - p$ by $p + 1$, $O$ is $p$ by $n - p - 1$, and $A_1$ and $A_2^T$ are vertex-edge incidence matrices of trees.

- The face $\mathcal{F}(A)$ of $\Omega_n$ is a simplex.
Property $N$ and contraction

Property $N$: For each pair $X_1$ and $X_2$ of complementary square submatrices of $A$,\[ PAQ = \begin{bmatrix} X_1 & * \\ * & X_2 \end{bmatrix}, \]
per $X_1 \leq 1$ or per $X_2 \leq 1$. 
Property $N$ and contraction

Property $N$: For each pair $X_1$ and $X_2$ of complementary square submatrices of $A$,

$$PAQ = \begin{bmatrix} X_1 & * \\ * & X_2 \end{bmatrix},$$

per $X_1 \leq 1$ or per $X_2 \leq 1$.

Contraction:

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \\ 0 & B \\ 0 & \vdots \end{pmatrix} \rightarrow \begin{pmatrix} \alpha + \beta \\ B \end{pmatrix}.$$
There exists an integer $p$ with $1 \leq p \leq n - 2$ and permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} A_3 & A_1 \\ A_2 & O \end{bmatrix}$$

where $A_3$ is $n - p$ by $p + 1$, $O$ is $p$ by $n - p - 1$, and $A_1$ and $A_2^T$ are vertex-edge incidence matrices of trees.
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Each 1 in $A_3$ is on a unique nonzero diagonal, since deleting a row of the vertex-incidence matrix of a tree leaves a square matrix with a unique nonzero diagonal.
Special Form

There exists an integer \( p \) with \( 1 \leq p \leq n - 2 \) and permutation matrices \( P \) and \( Q \) such that

\[
P AQ = \begin{bmatrix}
A_3 & A_1 \\
A_2 & O
\end{bmatrix}
\]

where \( A_3 \) is \( n - p \) by \( p + 1 \), \( O \) is \( p \) by \( n - p - 1 \), and \( A_1 \) and \( A_2^T \) are vertex-edge incidence matrices of trees.

Each \( 1 \) in \( A_3 \) is on a unique nonzero diagonal, since deleting a row of the vertex-incidence matrix of a tree leaves a square matrix with a unique nonzero diagonal.

Remark: A matrix of this form is fully indecomposable iff it has at least two 1’s in each row and column.
The face $\mathcal{F}(A)$ of $\Omega_n$ is a simplex.
Simplex

- The face \( \mathcal{F}(A) \) of \( \Omega_n \) is a simplex.

- For each pair of distinct permutation matrices \( P, Q \leq A \), 
  \[ P^{-1}Q \] has at most one (permutation) cycle of length \( \geq 2 \).
Simplex

- The face $\mathcal{F}(A)$ of $\Omega_n$ is a simplex.

- For each pair of distinct permutation matrices $P, Q \leq A$, $P^{-1}Q$ has at most one (permutation) cycle of length $\geq 2$.

Remark: The only other faces of $\Omega_n$ with this property are $\mathcal{F}(A)$ where $A$ either equals $J_3$ or can be contracted to $J_3$. 

Some equivalences

RAB and P. Gibson (1977) showed that \textit{simplex face}, \textit{property} $N$ \textit{and non-contractibility to} $J_3$, \textit{and special form} are equivalent.
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The special form

\[
PAQ = \begin{bmatrix}
A_3 & A_1 \\
A_2 & O
\end{bmatrix}
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where \( A_3 \) is \( n-p \) by \( p+1 \), \( O \) is \( p \) by \( n-p-1 \), and \( A_1 \) and \( A_2^T \) are vertex-edge incidence matrices of trees. implies that there is a 1-1 correspondence between nonzero diagonals of \( A \) and the 1’s in \( A_3 \) and hence \( D(A) \) is isolated.
Some equivalences continued

Not Property $N$ implies that there exist $X_1$ and $X_2$, complementary square submatrices of $A$, such that $\text{per } X_1 \geq 2$ and $\text{per } X_2 \geq 2$. This gives 4 permutation matrices $P \leq A$ none of which contains a 1 not in any other nonzero diagonal. So $\mathcal{D}(A)$ is not isolated.
Some equivalences continued

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Since \( J_3 \) has the property that every 1 is on two nonzero diagonals, every matrix contractable to \( J_3 \) also has this property, and so cannot have an isolated set of diagonals.
Some equivalences continued

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Thus we have the equivalences:

Characterization III of $\mathcal{D}(A)$ isolated

Let $A$ be a fully indecomposable matrix of order $n \geq 2$. Then the following are equivalent:

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- $A$ has property $N$, and cannot be contracted to $J_3$.
- There exists an integer $p$ with $1 \leq p \leq n - 2$ and permutation matrices $P$ and $Q$ such that
  $$PAQ = \begin{bmatrix}
A_3 & A_1 \\
A_2 & O
\end{bmatrix}$$
  where $A_3$ is $n - p$ by $p + 1$, $O$ is $p$ by $n - p - 1$, and $A_1$ and $A_2^T$ are vertex-edge incidence matrices of trees.
- The face $\mathcal{F}(A)$ of $\Omega_n$ is a simplex.
Recall: If $I_n \leq A$, then $D(A)$ is an isolated set of diagonals iff the following two conditions are satisfied:

- In the digraph $\Gamma(A)$ of $A$ there is a vertex that belongs to all directed cycles of length $> 1$.
- $\Gamma(A)$ is free-cyclic meaning that each cycle contains an edge that is not an edge of any other directed cycle.
Digraph characterization

Recall: If $I_n \leq A$, then $\mathcal{D}(A)$ is an isolated set of diagonals iff the following two conditions are satisfied:

1. In the digraph $\Gamma(A)$ of $A$ there is a vertex that belongs to all directed cycles of length $> 1$.

2. $\Gamma(A)$ is free-cyclic meaning that each cycle contains an edge that is not an edge of any other directed cycle.

Under the assumption that $I_n \leq A$, it is fairly straightforward to check that these two conditions are equivalent to $\mathcal{D}(A)$ being isolated.
Digraph characterization

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Under the assumption that $I_n \leq A$, it is fairly straightforward to check that these two conditions are equivalent to $D(A)$ being isolated.

What is the connection between the special form and this digraph characterization?
Special form and digraph

Special form implies that $A$ has a zero submatrix $O_{p,n-p-1}$. With the assumption that $I_n \leq A$, the row indices and column indices of $O_{p,n-p-1}$ must be disjoint, and by simultaneous permutations (so digraph is the same), we can assume

$$A = \begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix}$$

($A_2$ is $p \times p + 1$, $A_1$ is $n - p \times n - p - 1$).
Special form and digraph

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$$A = \begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix} \quad (A_2 \text{ is } p \times p + 1, \ A_1 \text{ is } n - p \times n - p - 1).$$

$A_3$ has a 1 in its upper right corner (position $(p + 1, p + 1)$ of $A$). Crossing out its row and column (so crossing out last column of $A_2$ and first row of $A_1$) gives $A_2'$ (order $p$) and $A_1'$ (order $n - p - 1$) with $I_p \leq A_2'$ and $I_{n-p-1} \leq A_1'$. 
\[ A = \begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix} \quad (A_2 \text{ is } p \times p + 1, \ A_1 \text{ is } n - p \times n - p - 1). \]
\[ A = \begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix} \] (\( A_2 \) is \( p \times p + 1 \), \( A_1 \) is \( n-p \times n-p-1 \)).

\( A_1 \) and \( A_2^T \) being vertex-edge incidence matrices of trees, \( A_2' \) and \( A_1' \) can be assumed to be triangular. Thus each cycle of the digraph of \( A \) contains vertex \( p + 1 \).
Special form and digraph continued

\[ A = \begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix} \]  
\((A_2 \text{ is } p \times p + 1, A_1 \text{ is } n - p \times n - p - 1)\).

\(A_1\) and \(A_2^T\) being vertex-edge incidence matrices of trees, \(A_2'\) and \(A_1'\) can be assumed to be triangular. Thus each cycle of the digraph of \(A\) contains vertex \(p + 1\).

Also each 1 in \(A_3\) other than the 1 (corresponds to an edge) on the main diagonal is in a unique nonzero diagonal and this edge is in a unique cycle.
\[ A = \begin{bmatrix} A_2 & O_{p,n-p-1} \\ A_3 & A_1 \end{bmatrix} \]  
\((A_2 \text{ is } p \times p + 1, A_1 \text{ is } n - p \times n - p - 1).\)

\(A_1\) and \(A_2^T\) being vertex-edge incidence matrices of trees, \(A_2'\) and \(A_1'\) can be assumed to be triangular. Thus each cycle of the digraph of \(A\) contains vertex \(p + 1\).

Also each 1 in \(A_3\) other than the 1 (corresponds to an edge) on the main diagonal is in a unique nonzero diagonal and this edge is in a unique cycle.

Thus the digraph is free-cyclic and there is a vertex belonging to all cycles of length \(\geq 2\).
Example

\[
\begin{bmatrix}
A_2 & O_{p,n-p-1} \\
A_3 & A_1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\( (n = 8, p = 3) \)
Some references


4. R.A. Brualdi and J.-Y. Shao, Isolated sets of diagonals, diagonal hypergraphs, and simplices of doubly stochastic matrices, submitted