The Bruhat Order for (0,1)-Matrices
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III. Bruhat orders on $A(R, S)$
Outline

I. Bruhat order on $S_n$

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Part I. Bruhat order on $S_n$
Definition of Bruhat order

$S_n$ denotes the set of permutations $\pi$ of $\{1, 2, \ldots, n\}$, equivalently the set of permutation matrices of order $n$. 
Definition of Bruhat order

\( \mathcal{S}_n \) denotes the set of permutations \( \pi \) of \( \{1, 2, \ldots, n\} \), equivalently the set of permutation matrices of order \( n \).

If \( \pi \) and \( \tau \) are in \( \mathcal{S}_n \), then

\[
\pi \preceq_B \tau
\]

provided \( \pi \) can be obtained from \( \tau \) by a sequence of inversion-reducing transpositions of the form

\[
(i_1, \ldots, i_k, \ldots, i_l, \ldots, i_n) \rightarrow (i_1, \ldots, i_l, \ldots, i_k, \ldots, i_n)
\]

where \( i_k > i_l \).
Definition of Bruhat order

$S_n$ denotes the set of permutations $\pi$ of $\{1, 2, \ldots, n\}$, equivalently the set of permutation matrices of order $n$. If $\pi$ and $\tau$ are in $S_n$, then

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provided $\pi$ can be obtained from $\tau$ by a sequence of inversion-reducing transpositions of the form

$$(i_1, \ldots, i_k, \ldots, i_l, \ldots, i_n) \rightarrow (i_1, \ldots, i_l, \ldots, i_k, \ldots, i_n)$$

where $i_k > i_l$.

Example ($n = 5$):

$$(3, 5, 4, 1, 2) \rightarrow (3, 1, 4, 5, 2) \rightarrow (2, 1, 4, 5, 3).$$
In terms of Permutation Matrices
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An inversion-reducing transposition replaces a submatrix of order 2 equal to $L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

$L_2 \rightarrow I_2$. 
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$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$ 

$$3, 5, 4, 1, 2 \rightarrow 3, 1, 4, 5, 2.$$
Basic Properties
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1. Partial order graded by the number of inversions.
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2. The identity permutation \((1, 2, \ldots, n)\) (with no inversions) is the unique minimal permutation and the anti-identity permutation \((n, n - 1, \ldots, 2, 1)\) (with \(n(n - 1)/2\) inversions) is the unique maximal permutation.
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Example:

$$L_4 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}$$
Basic Properties continued
Cover relation: If $P_1$ is obtained from $P_2$ by an $L_2 \rightarrow I_2$ interchange, then $P_2$ covers $P_1$ (or $P_1$ is covered by $P_2$) in the Bruhat order provided the submatrix of consecutive rows and columns “spanned” by the $L_2$-submatrix of $P_2$ contains no other 1’s.
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For example,

\[
P_2 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\text{ covers }
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
Cover relation: If $P_1$ is obtained from $P_2$ by an $L_2 \rightarrow I_2$ interchange, then $P_2$ covers $P_1$ ( $P_1$ is covered by $P_2$) in the Bruhat order provided the submatrix of consecutive rows and columns “spanned” by the $L_2$-submatrix of $P_2$ contains no other 1’s.

For example,

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$ covers $P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

$4, 5, 1, 6, 2, 3$ covers $4, 2, 1, 6, 5, 3$
Equivalent ways to define Bruhat order
Equivalent ways to define Bruhat order

\[ \sigma = (i_1, i_2, \ldots, i_n) \text{ and } \tau = (j_1, j_2, \ldots, j_n) \] permutations of \( \{1, 2, \ldots, n\} \). For each \( k \) with \( 1 \leq k \leq n - 1 \), let \( i_{1k}, i_{2k}, \ldots, i_{kk} \) be the increasing rearrangement of \( i_1, i_2, \ldots, i_k \), with \( j_{1k}, j_{2k}, \ldots, j_{kk} \) defined in a similar way.
Equivalent ways to define Bruhat order

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\( \sigma \preceq_B \tau \) if and only if \( i_{pk} \leq j_{pk} \) for all \( p \) and \( k \) with \( 1 \leq p \leq k \leq n - 1 \).
Equivalent ways to define Bruhat order

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\( \sigma \preceq_B \tau \) if and only if \( i_{pk} \leq j_{pk} \) for all \( p \) and \( k \) with \( 1 \leq p \leq k \leq n - 1 \).

Example: if \( \sigma = (2, 1, 4, 5, 3) \) and \( \tau = (3, 1, 5, 4, 2) \), then \( \sigma \preceq_B \tau \) because of the entrywise inequalities satisfied by the arrays

\[
\begin{bmatrix}
1 & 2 & 4 & 5 \\
1 & 2 & 4 & \\
1 & 2 & \\
2 & \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 4 & 5 \\
1 & 3 & 5 \\
1 & 3 \\
3 & \\
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix}
\]
In terms of matrices ...
In terms of matrices ...

For an \( m \) by \( n \) matrix \( A = [a_{ij}] \), let

\[
\sigma_{ij}(A) = \sum_{k=1}^{i} \sum_{l=1}^{j} a_{kl} \quad (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n),
\]

the sum of the entries of \( A \) in its leading \( i \) by \( j \) submatrix.
In terms of matrices ...

For an $m$ by $n$ matrix $A = [a_{ij}]$, let

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the sum of the entries of $A$ in its leading $i$ by $j$ submatrix.

Let

$$\Sigma(A) = [\sigma_{ij}(A); i = 1, 2, \ldots, m; j = 1, 2, \ldots, n].$$
In terms of matrices ...

For an \( m \) by \( n \) matrix \( A = [a_{ij}] \), let

\[
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\]

the sum of the entries of \( A \) in its leading \( i \) by \( j \) submatrix.

Let

\[
\Sigma(A) = [\sigma_{ij}(A); i = 1, 2, \ldots, m; j = 1, 2, \ldots, n].
\]

Then for permutation matrices \( P \) and \( Q \) of order \( n \),

\[
P \preceq_B Q \text{ if and only if } \Sigma(P) \geq \Sigma(Q) \text{ (entrywise).}
\]
II. $\mathcal{A}(R, S)$
Definition of $\mathcal{A}(R, S)$
Definition of $\mathcal{A}(R, S)$

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be positive integral vectors. Then $\mathcal{A}(R, S)$ denotes the set of all $(0, 1)$-matrices with row sum vector $R$ and column sum vector $S$. 
Definition of $\mathcal{A}(R, S)$

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be positive integral vectors. Then $\mathcal{A}(R, S)$ denotes the set of all $(0, 1)$-matrices with row sum vector $R$ and column sum vector $S$.

**Example:** The matrix

$$A = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1
\end{bmatrix}$$

belongs to $\mathcal{A}(R, S)$ where $R = (3, 2, 3, 4)$ and $S = (3, 2, 3, 2, 2)$. 
Two important properties of $A(R, S)$
Two important properties of $\mathcal{A}(R, S)$

Existence: Gale-Ryser theorem gives necessary and sufficient conditions for $\mathcal{A}(R, S)$ to be nonempty.
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“Connectivity”: Given $A_1, A_2 \in \mathcal{A}(R, S)$, $A_1$ can be transformed into $A_2$ by a sequence of interchanges:

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Each of which replaces a submatrix of $A_1$ equal to $L_2$ with $I_2$, or the other way around.
Two important properties of $\mathcal{A}(R, S)$

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each of which replaces a submatrix of $A_1$ equal to $L_2$ with $I_2$, or the other way around.

III. Bruhat orders on $A(R, S)$
Two possibilities
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\[(B) \quad (\text{Bruhat order on } \mathcal{A}(R, S)) \quad A_1 \preceq_B A_2 \quad \text{provided that} \quad \Sigma(A_1) \geq \Sigma(A_2).\]
Bruhat orders on a nonempty $\mathcal{A}(R, S)$

Two possibilities

(B) *(Bruhat order on $\mathcal{A}(R, S)$)* $A_1 \preceq_B A_2$ provided that $\Sigma(A_1) \geq \Sigma(A_2)$.

(\(\hat{B}\)) *(Secondary Bruhat order on $\mathcal{A}(R, S)$)* $A_1 \preceq_{\hat{B}} A_2$ provided that $A_1$ can be obtained from $A_2$ by a sequence of $L_2 \rightarrow I_2$ interchanges.
Bruhat orders on a nonempty $\mathcal{A}(R, S)$

Two possibilities

(B) (Bruhat order on $\mathcal{A}(R, S)$) $A_1 \leq_B A_2$ provided that $\Sigma(A_1) \geq \Sigma(A_2)$.

(\(\hat{B}\)) (Secondary Bruhat order on $\mathcal{A}(R, S)$) $A_1 \leq_{\hat{B}} A_2$ provided that $A_1$ can be obtained from $A_2$ by a sequence of $L_2 \to I_2$ interchanges.

Note that if $A_1$ is obtained from $A_2$ by an $L_2 \to I_2$ interchange, then $\Sigma(A_1) \geq \Sigma(A_2)$, that is the Bruhat order is a refinement of the secondary Bruhat order.
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$(B)$ (Bruhat order on $\mathcal{A}(R, S)$) $A_1 \preceq_B A_2$ provided that $\Sigma(A_1) \geq \Sigma(A_2)$.

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Note that if $A_1$ is obtained from $A_2$ by an $L_2 \rightarrow I_2$ interchange, then $\Sigma(A_1) \geq \Sigma(A_2)$, that is the Bruhat order is a refinement of the secondary Bruhat order.

Are these two partial orders the same as they are on permutation matrices? This was implicitly conjectured to be so by RAB and Hwang.
Secondary Bruhat order cover relation
Let \( A = [a_{ij}] \) be a matrix in \( A(R, S) \) where \( A\{i, j\}, \{k, l\} = L_2 \).

Let \( A' = [a'_{ij}] \) be the matrix obtained from \( A \) by the \( L_2 \rightarrow I_2 \) interchange that replaces \( A\{i, j\}, \{k, l\} = L_2 \) with \( I_2 \).
Secondary Bruhat order cover relation

Let $A = [a_{ij}]$ be a matrix in $\mathcal{A}(R, S)$ where $A[\{i, j\}, \{k, l\}] = L_2$. Let $A' = [a'_{ij}]$ be the matrix obtained from $A$ by the $L_2 \rightarrow I_2$ interchange that replaces $A[\{i, j\}, \{k, l\}] = L_2$ with $I_2$. Then $A$ covers $A'$ in the secondary Bruhat order on $\mathcal{A}(R, S)$ if and only if

(i) $a_{pk} = a_{pl}$, $(i < p < j)$, and $a_{iq} = a_{jq}$, $(k < q < l)$

\[
\begin{bmatrix}
0 & a_{iq} & 1 \\
0 & 1 & a_{jq} \\
1 & a_{iq} & 0 \\
a_{pk} & a_{pl} & 1
\end{bmatrix}
\]

(ii) $a_{pk} = 0$ and $a_{iq} = 0$ imply $a_{pq} = 0$, $(i < p < j, k < q < l)$,

(iii) $a_{pk} = 1$ and $a_{iq} = 1$ imply $a_{pq} = 1$, $(i < p < j, k < q < l)$. 

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The Conjecture
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First recall: $A_1 \preceq_B A_2$ implies that $A_1 \preceq_B A_2$, that is, the Bruhat order is a refinement of the secondary Bruhat order.
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Let \( \mathcal{A}(n, k) \) denote the class of all \((0, 1)\)-matrices with \( k \) 1’s in each row and column. This class is nonempty for all \( 0 \leq k \leq n \).
The Conjecture

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$A(n, 1)$ is the class of permutation matrices of order $n$. 
The Conjecture

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Let $A(n, k)$ denote the class of all $(0, 1)$-matrices with $k$ 1’s in each row and column. This class is nonempty for all $0 \leq k \leq n$.

$A(n, 1)$ is the class of permutation matrices of order $n$.

$A(n, 2)$ is the class of $(0, 1)$-matrices of order $n$ that can be written as the sum of two permutation matrices.
Three matrices in $\mathcal{A}(6, 3)$

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix},
C = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.
\]
And their $\Sigma$ matrices

$$\Sigma_A = \begin{bmatrix}
1 & 1 & 1 & 1 & 2 & 3 \\
2 & 2 & 3 & 4 & 5 & 6 \\
3 & 4 & 5 & 7 & 8 & 9 \\
3 & 4 & 5 & 8 & 10 & 12 \\
3 & 5 & 7 & 10 & 12 & 15 \\
3 & 6 & 9 & 12 & 15 & 18 \\
\end{bmatrix}, \quad \Sigma_C = \begin{bmatrix}
0 & 0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 4 & 5 & 6 \\
2 & 3 & 4 & 7 & 8 & 9 \\
3 & 4 & 5 & 8 & 10 & 12 \\
3 & 5 & 7 & 10 & 13 & 15 \\
3 & 6 & 9 & 12 & 15 & 18 \\
\end{bmatrix},$$

$$\Sigma_D = \begin{bmatrix}
0 & 0 & 0 & 1 & 2 & 3 \\
1 & 2 & 2 & 4 & 5 & 6 \\
2 & 3 & 4 & 7 & 8 & 9 \\
3 & 4 & 5 & 8 & 10 & 12 \\
3 & 5 & 7 & 10 & 13 & 15 \\
3 & 6 & 9 & 12 & 15 & 18 \\
\end{bmatrix}.$$
And their relationships
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Comparing, we have $\Sigma_A > \Sigma_D > \Sigma_C$ (entrywise)
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And so $A \prec_B D \prec_B C$.  (Bruhat order)
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And so $A \preccurlyeq_B D \preccurlyeq_B C$. (Bruhat order)

How do $A, C, D$ compare in the secondary Bruhat order?
Comparing, we have $\sum_A > \sum_D > \sum_C$ (entrywise).

And so $A \prec_B D \prec_B C$. (Bruhat order)

How do $A, C, D$ compare in the secondary Bruhat order?

From the cover relation for the secondary Bruhat order on classes $\mathcal{A}(R, S)$, we get that $C$ covers both $A$ and $D$ in the secondary Bruhat order, implying that $A$ and $D$ are incomparable in the secondary Bruhat order.

Thus $\preceq_B$ and $\preceq_{\overline{B}}$ are different on $\mathcal{A}(6, 3)$, and the Bruhat order is a proper refinement of the Bruhat order on $\mathcal{A}(6, 3)$ and on $\mathcal{A}(R, S)$ in general.
Two Theorems
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Theorem: On $A(n, 2)$ the Bruhat order and secondary Bruhat order are identical.
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Theorem: On $\mathcal{A}(n, 2)$ the Bruhat order and secondary Bruhat order are identical.

The class $\mathcal{A}(n, 1)$ (and so the complementary class $\mathcal{A}(n, n - 1)$) contains a unique minimal matrix in the (secondary) Bruhat order. So do the trivial complementary classes $\mathcal{A}(n, 0)$ and $\mathcal{A}(n, n)$ each of cardinality 1.
Two Theorems

Theorem: On $\mathcal{A}(n, 2)$ the Bruhat order and secondary Bruhat order are identical.

The class $\mathcal{A}(n, 1)$ (and so the complementary class $\mathcal{A}(n, n - 1)$) contains a unique minimal matrix in the (secondary) Bruhat order. So do the trivial complementary classes $\mathcal{A}(n, 0)$ and $\mathcal{A}(n, n)$ each of cardinality 1.

Theorem: The only other classes $\mathcal{A}(n, k)$ that contain a unique minimal matrix in the secondary Bruhat order are the classes $\mathcal{A}(2k, k)$; the unique minimal matrix is $J_{k} \oplus J_{k}$. 
Algorithm to construct a minimal matrix
Algorithm to construct a minimal matrix

An algorithm to construct a minimal matrix in general classes \( A(\mathbb{R}, \mathbb{S}) \) is given by RAB and Hwang.
Algorithm for $A(n, k)$
Algorithm for $A(n, k)$

1. Let $n = qk + r$ where $0 \leq r < k$.

2. If $r = 0$, then $A = J_k \oplus \cdots \oplus J_k$, ($q$ $J_k$’s) is a minimal matrix.

3. Else, $r \neq 0$.
   
   (a) If $q \geq 2$, let
   
   $$A = X \oplus J_k \oplus \cdots \oplus J_k, \; (q - 1 \; J_k \; \text{'s}, \; X \; \text{has order} \; k + r),$$
   
   and let $n \leftarrow k + r$.
   
   (b) Else, $q = 1$, and let
   
   $$A = \begin{bmatrix} J_{r,k} & O_k \\ X & J_{k,r} \end{bmatrix}, \; (X \; \text{has order} \; k),$$
   
   and let $n \leftarrow k$ and $k \leftarrow k - r$.

   (c) Proceed recursively with the current values of $n$ and $k$ to determine $X$. 
Constructed minimal matrix for $A(18, 11)$
Constructed minimal matrix for $\mathcal{A}(18, 11)$

$$
\begin{bmatrix}
  J_{7,11} & O_7 \\
  J_{3,4} & O_3 & O_{7,4} \\
  I_4 & J_{4,3} & J_{11,7} \\
  O_{4,7} & J_4
\end{bmatrix}
$$
**Constructed minimal matrix for \(A(18, 11)\)**

\[
\begin{bmatrix}
    J_{7,11} \\
    J_{3,4} & O_3 \\
    I_4 & J_{4,3} \\
    O_{4,7} & J_4 \\
\end{bmatrix}
\]

The minimal matrices in \(A(n, 2)\) and \(A(n, 3)\) have been characterized, but there does not appear to be a useful characterization of the minimal matrices in \(A(n, k)\) for \(k \geq 4\).
IV. Conclusion/Open Problems
Conclusions
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The Bruhat order on permutations extends in general in two ways to a Bruhat order on classes $A(R, S)$. 
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The Bruhat order on permutations extends in general in two ways to a Bruhat order on classes $\mathcal{A}(R, S)$.

While there are some similarities, the structures on classes $\mathcal{A}(R, S)$, even regular classes $\mathcal{A}(n, k)$, are, not surprisingly, much more complicated.
Conclusions

The Bruhat order on permutations extends in general in two ways to a Bruhat order on classes $\mathcal{A}(R, S)$.

While there are some similarities, the structures on classes $\mathcal{A}(R, S)$, even regular classes $\mathcal{A}(n, k)$, are, not surprisingly, much more complicated.

The Bruhat order on permutations $\mathcal{A}(n, 1)$ is graded, but this is not the case for classes $\mathcal{A}(R, S)$, even classes $\mathcal{A}(n, k)$. 
Open problems
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2. A characterization of the cover relation of the secondary Bruhat order on classes $A(R, S)$ has been determined. Is there a nice characterization of the cover relation of the Bruhat order on $A(R, S)$?
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4. The definition of the Bruhat order on \( A(R, S) \) provides an efficient method to check whether \( A_1 \preceq A_2 \) via the matrices \( \Sigma_{A_1} \) and \( \Sigma_{A_2} \). Is there an efficient way to check whether \( A_1 \preceq_B A_2 \)?