Hankel operators and VMO on the bi-disc

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Abstract. We first extend Nehari's theorem concerning the boundedness of Hankel operators of one complex variable to the bi-disc with a product dilation structure. This fact relies on a result of S. Ferguson and M. Lacey [8] which gives a way to characterize the boundedness of the *little* Hankel operators of two complex variables in terms of the product BMO space of S.-Y. Chang and R. Fefferman [2,3]. The main results of the paper then give several characterizations of the compactness of the little Hankel operators. The first is in terms of the product VMO space, and the others are an extension of Hartman's theorem to the product domain. In addition, we phrase these results in terms of commutators and Carleson measures. Finally, we obtain the two parameter duality result $VMO^* = H^1$. These results are joint work with M. Lacey and B. Wick, first published in the Proceedings of the American Mathematical Society in Volume 134, Number 2.

1. Overview of One Parameter Theory

We begin with a brief survey of the one parameter theory. By this, we mean that the families of functions and operators involved are invariant under a one parameter dilation, whether the functions are defined on one or several variables. There are several function spaces that are critical to the study of Hankel operators, the first of which are the Hardy spaces $H^p(\mathbb{D})$. For $1 \le p \le \infty$, $H^p(\mathbb{D})$ is the space consisting of analytic functions F in $\mathbb{D}$ such that

$$\|F\|_p := \sup_{0<r<1} \left(\int_{\mathbb{T}} |F(re^{2\pi i\theta}| \, d\theta \right)^{1/p} < \infty.$$

Since F has nontangential boundary values almost everywhere on $\mathbb{T}$, we can identify $H^p(\mathbb{D})$ with the space of its boundary values. $H^p(\mathbb{D})$ can also be seen as the closure in $L^p(\mathbb{T})$ norm of the span of the exponentials $\{z^n : n \in \mathbb{N}\}$. In other words, for $f \in L^p(\mathbb{T})$, if the extension $F(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ is analytic in the disc, we regard f as an $H^p(\mathbb{D})$ function.

2000 *Mathematics Subject Classification.* Primary 47B35, 32A37; Secondary 32A35, 42B35.
Key words and phrases. Hankel operators, product VMO, product BMO, polydisc.
The author's research supported in part by an NSF VIGRE grant to the Georgia Institute of Technology.

John and Nirenberg [13] first introduced the space of functions of bounded mean oscillation, or BMO, in 1961. For $f \in L^1(\mathbb{T})$ and I a subarc of $\mathbb{T}$,

$$f \in BMO(\mathbb{T}) \text{ if and only if } \sup_I \frac{1}{m(I)} \int_I |f - f_I|\, dm < \infty,$$

where m denotes the normalized Lebesgue measure on $\mathbb{T}$ and f_I is the mean value of f over I. In 1975, Sarason [19] described a subspace of BMO called the space of vanishing mean oscillation, or VMO. We say that $f \in VMO(\mathbb{T})$ if and only if

$$\lim_{m(I)\to 0} \frac{1}{m(I)} \int_I |f - f_I|\, dm = 0,$$

which is equivalent to saying f is in the closure of $C(\mathbb{T})$ with respect to the BMO topology.

The definitions and results above can also be phrased on $\mathbb{T}^n$ or on the real line and $\mathbb{R}^n$ via a conformal mapping. We can identify $H^p(\mathbb{C}^n_+)$ with the space of functions that have a holomorphic extension to the upper half space

$$\mathbb{C}^n_+ = \prod_{j=1}^{n} \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}.$$

We will especially be interested in the real Hardy spaces, denoted $H^p(\mathbb{R}^n)$, which consist of the real parts of the boundary values of functions in $H^p(\mathbb{C}^n_+)$. There are several equivalent norms on this space in terms of square functions, maximal functions, and the Hilbert transform in one dimension or the Riesz transforms in $n > 1$ dimensions. See the seminal paper of C. Fefferman and E. M. Stein [6] for details. In the early 70's, C. Fefferman [5] and C. Fefferman and E. M. Stein [6] proved the remarkable result that $BMO(\mathbb{R}^n)$ is the dual of the Hardy space $H^1(\mathbb{R}^n)$. This was done by showing a certain Carleson measure condition. This Carleson measure cannot simply be tensored up to obtain a multi-parameter duality theorem, as we will see later. On the disc, $BMO(\mathbb{T})$ is the dual of $Re(H^1(\mathbb{D}))$ which is the subspace of functions in $L^1(\mathbb{T})$ whose Hilbert transform is also in $L^1(\mathbb{T})$.

Hankel operators are an important tool used in the study of such problems as the Nevanlinna–Pick interpolation problem. To define these operators we first notice that

$$L^2(\mathbb{T}) = H^2(\mathbb{D}) \oplus \overline{H^2(\mathbb{D})},$$

where $H^2(\mathbb{D})$ and $\overline{H^2(\mathbb{D})}$ are the space of square integrable functions with respectively analytic and anti-analytic extensions in $\mathbb{D}$. We can also think of $H^2(\mathbb{D})$ as the space of square integrable functions with vanishing Fourier coefficients corresponding to the negative frequencies and vice versa for $\overline{H^2(\mathbb{D})}$. Define the Riesz projection

$$\mathbb{P}_- : L^2(\mathbb{T}) \to \overline{H^2(\mathbb{D})}.$$

For $\varphi \in L^2(\mathbb{T})$, often called the symbol, we define the Hankel operator

$$H_\varphi : H^2(\mathbb{D}) \longrightarrow \overline{H^2(\mathbb{D})},$$

$$H_\varphi f = \mathbb{P}_- M_\varphi f,$$

where M_φ denotes pointwise multiplication by φ. A well-known theorem of Nehari describes the boundedness of the Hankel operators in terms of the symbol.

1.1. Theorem. *(Nehari [16]) H_φ is bounded if and only if there is a function $\psi \in L^\infty(\mathbb{T})$ for which*

$$\mathbb{P}_-\varphi = \mathbb{P}_-\psi.$$

In this case,

$$\begin{aligned}\|H_\varphi\| =& \inf\{\|\psi\|_\infty : \widehat{\varphi}(m) = \widehat{\psi}(m), \quad m < 0\} \\ =& \|\mathbb{P}_-\varphi\|_{BMO(\mathbb{T})}.\end{aligned}$$

The classical Nehari problem deals with approximating an L^∞ function by bounded, analytic functions. If $\varphi \in L^\infty$, then Nehari's theorem says that

$$\|H_\varphi\| = \inf\{\|\varphi - f\|_\infty : f \in H^\infty\}.$$

Thus the norm of the Hankel operator can viewed as the distance from φ to H^∞.

The original theorem of Nehari came before the introduction of BMO and so did not have this aspect of the above theorem. In his original proof, Nehari used the crucial idea that each $H^1(\mathbb{D})$ function can be written as a product of $H^2(\mathbb{D})$ functions with equality of norms. This factorization does not carry over to several variables let alone several parameters. In 1976, R. Coifman, R. Rochberg, and G. Weiss proved a "weak factorization" result for functions in $H^1(\mathbb{R}^n)$. Namely

$$f \in H^1(\mathbb{R}^n) \text{ iff } \exists (g_j), (h_j) \in H^2(\mathbb{R}^n) \text{ s.t. } f = \sum_j g_j h_j \text{ and } \sum_j \|g_j\|_2 \|h_j\|_2 < \infty.$$

Therefore, if we relax equality to equivalence, then Nehari's Theorem extends to $\mathbb{T}^n$. Once $H^1 - BMO$ duality was established, another proof of Nehari's theorem became available. This uses C. Fefferman's [5] result that

$$BMO = \{\xi + \widetilde{\eta} : \xi, \eta \in L^\infty\},$$

where $\widetilde{\eta}$ denotes the harmonic conjugate of η. On the real line, the harmonic conjugate is replaced by the Riesz projection of L^2 onto H^2. In either case, one can readily see that if the projection of φ onto its negative Fourier coefficients is in BMO, then it agrees with an L^∞ function on the negative Fourier coefficients.

We now consider the issue of compactness of Hankel operators. We first have the following Hartman compactness criteria. Let $H^\infty + C$ denote the functions $\varphi \in L^\infty$ such that $\varphi = f + g$, where $f \in H^\infty$ and $g \in C(\mathbb{T})$.

1.2. Theorem. *(Hartman [12]) Let $\varphi \in L^\infty$. TFAE*

(i) *H_φ is compact.*
(ii) *$\varphi \in H^\infty + C$.*
(iii) *$\exists\, \psi \in C(\mathbb{T})$ such that $H_\varphi = H_\psi$.*

We note that the condition that φ be an L^∞ function is only necessary for part (ii) above. There is another characterization of compact Hankel operators in terms of the space VMO.

1.3. Theorem. *(Sarason [18, 19]) Let $\varphi \in L^2$. Then H_φ is compact if and only if $\mathbb{P}_-\varphi \in VMO$.*

This follows easily from Theorem 1.2 once we have the additional facts, due to Sarason, that $H^\infty + C$ is a closed subalgebra of L^∞ and

$$VMO = \{\xi + \widetilde{\eta} : \xi, \eta \in C(\mathbb{T})\}.$$

In two or more parameters, there is no such factorization which adds a level of difficulty to the problem.

Throughout, by $A \lesssim B$ we mean that there is an absolute constant K so that $A \leq KB$. By $A \approx B$ we mean $A \lesssim B$ and $B \lesssim A$.

2. The Hardy and BMO Spaces on the Bi-disc

In product theory on the polydisc, one is generally concerned with operators acting on functions on $\mathbb{D}^{n_1} \times \mathbb{D}^{n_2} \times \cdots \times \mathbb{D}^{n_k}$ invariant under a k-parameter family of dilations defined by

$$(x_1, x_2, \ldots, x_k) \rightarrow (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_k x_k).$$

Operators on functions of complex and real product spaces are also commonly investigated. We seek to extend the ideas in Section 1 to two parameters. Namely, we'll consider product spaces like $\mathbb{D} \otimes \mathbb{D}$ where we allow dilations independent in each variable. The structure in this setting is quite different and is not merely a trivial extension of the one parameter case.

The analytic product Hardy spaces $H^p(\mathbb{D} \otimes \mathbb{D})$ consist of complex functions F analytic on the bi-disc in each variable separately for which

$$\|F\|_{H^p}^p := \sup_{0<r_j<1} \int_{\mathbb{T}\otimes\mathbb{T}} |F(r_1 e^{2\pi i\theta_1}, r_2 e^{2\pi i\theta_2})|^p \, d\theta_1 \, d\theta_2 < \infty.$$

The boundary values of F exist almost everywhere when $1 \leq p \leq \infty$, and so we again can identify $H^p(\mathbb{D}\otimes\mathbb{D})$ with the space of its boundary values. For functions in $H^p(\mathbb{D} \otimes \mathbb{D})$, the Fourier coefficients can be nonzero only on the positive frequencies in each variable. The analytic Hardy spaces $H^p(\mathbb{C}_+ \otimes \mathbb{C}_+)$ consist of complex functions F analytic in each variable separately such that

$$\|F\|_{H^p}^p := \sup_{y_1,y_2>0} \int_{\mathbb{R}^2} |F(x_1 + iy_1, x_2 + iy_2)|^p \, dx_1 dx_2 < \infty.$$

We also need the real Hardy space H^1. The real Hardy space $H^1(\mathbb{R} \otimes \mathbb{R})$ consists of real valued functions f on $\mathbb{R}^2$ for which

$$\|f\|_{H^1(\mathbb{R}\otimes\mathbb{R})} := \sum_{A_1,A_2 \in \{I,H_1,H_2\}} \|A_1 A_2 f\|_1 < \infty,$$

where I is the identity operator and H_j is the Hilbert transform computed in the jth coordinate. This defers from the n-dimensional one parameter case where the norm can be taken as

$$\|f\|_{H^1(\mathbb{R}^n)} := \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)},$$

where R_j is the j-th Riesz transform. For $f \in H^1(\mathbb{R} \otimes \mathbb{R})$, there is a bianalytic extension $F(z_1, z_2)$ to $\mathbb{C}_+ \otimes \mathbb{C}_+$ such that

$$\lim_{y_1,y_2 \downarrow 0} \operatorname{Re} F(x_1 + iy_1, x_2 + iy_2) = f(x_1, x_2) \qquad \text{a.e.}$$

There are several equivalent definitions of this Hardy space in terms of maximal, square, and area functions in the product setting. See the papers of R. F. Gundy [10] and R. F. Gundy and E. M. Stein [11].

The dual space $BMO(\mathbb{R} \otimes \mathbb{R})$ of real $H^1(\mathbb{R} \otimes \mathbb{R})$ was identified by S.-Y. Chang and R. Fefferman [2, 3]. This is a nontrivial extension of the one parameter case since the associated Carleson measures are far more complicated. In addition, there is not a nice, intrinsic definition as the one given by John and Nirenberg in one parameter.

Although not an intrinsic definition, we can give another characterization of product BMO in terms of wavelets and Carleson measures which is very useful. Let $\mathcal{D}$ denote the set of dyadic intervals on $\mathbb{R}$. Given a rectangle $R = R_1 \times R_2 \in \mathcal{D} \times \mathcal{D}$ define translation and dilation invariant operators

$$\begin{aligned} T_y f(x) :=& f(x-y), \qquad y \in \mathbb{R}^2, \\ D^p_{R_1\times R_2} f(x_1,x_2) :=& \frac{1}{(|R_1||R_2|)^{1/p}} f\Big(\frac{x_1}{|R_1|}, \frac{x_2}{|R_2|}\Big), \qquad 0 < p < \infty. \end{aligned}$$

Note that the dilation operator preserves L^p norm and depends upon the scale but not location of the rectangle $R_1 \times R_2$. We let v be a function on $\mathbb{R}$ of mean value zero which is bounded, piecewise continuous, rapidly decreasing and satisfies the following. If we set

$$\begin{aligned} w(x_1,x_2) &= v(x_1)v(x_2), \text{ and} \\ w_R &= T_{c(R)} D^2_R w, \end{aligned}$$

where $c(R)$ is the center of R, then $\{w_R : R \in \mathcal{D}\times\mathcal{D}\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

The function v is a building block for a wavelet basis. We can build specific wavelets that will do the job. For example, one can take the Haar wavelets which come from dilations and translations of the product of the one variable function $h = -\mathbf{1}_{[0,1/2)} + \mathbf{1}_{[1/2,1]}$. Another example is a Meyer wavelet which is built up from a Schwartz function v of L^2 norm one whose Fourier transform is supported in some annulus, say $2\pi \le |\xi| \le 8\pi$. Notice that the Haar wavelets have compact spatial support but are not smooth, while the Meyer wavelet is smooth and does not have compact spatial support. It does, however, decay faster than the reciprocal of any polynomial.

In the language of wavelets, a theorem of Chang and Fefferman [2, 3] then becomes

$$\|f\|_{BMO(\mathbb{R}\otimes\mathbb{R})} \approx \sup_U \Big[|U|^{-1} \sum_{R\subset U} |\langle f, w_R\rangle|^2\Big]^{1/2}. \tag{2.1}$$

As in one parameter BMO, the "norm" above is actually a semi-norm, but we will consider it a norm by identifying constant functions with 0. An essential part of this definition is that the supremum be formed over all subsets U of the plane with finite measure. In one dimension, this is equivalent to

$$\|f\|_{BMO(\mathbb{R})} \approx \sup_J \Big[|J|^{-1} \sum_{I\subset J} |\langle f, w_I\rangle|^2\Big]^{1/2},$$

where J is any interval and I is a dyadic interval. If we merely take the tensor product of such an expression, then the supremum in (2.1) would be over rectangles U in the plane. However, doing so generates the space referred to as rectangular BMO, or $BMO(rec)$. This space is decidedly larger than BMO, as an example of Carleson [1] demonstrates. R. Fefferman [7] used Carleson's example to produce functions which act as linear functionals on $H^1(\mathbb{R}\otimes\mathbb{R})$ with norm one, yet have arbitrarily small $BMO(rec)$ norm.

As with the Hardy spaces, there are both analytic and real product BMO spaces. To define analytic $BMO(\mathbb{C}_+ \otimes \mathbb{C}_+)$, the definition (2.1) can still be used with the functions w_R replaced by the jointly analytic projections $\mathbb{P}_{+,+}w_R$.

3. Boundedness of the Hankel Operators on $H^2(\mathbb{D} \otimes \mathbb{D})$

Let us now define the *little* Hankel operators. We let $H^2_{\pm,\pm}(\mathbb{D} \otimes \mathbb{D})$ denote the space of square integrable functions with (anti) analytic extensions in each variable separately. $L^2(\mathbb{T} \otimes \mathbb{T})$ is then the direct sum of

$$L^2(\mathbb{T} \otimes \mathbb{T}) = \oplus_{\varepsilon \in \{\pm,\pm\}} H^2_\varepsilon(\mathbb{D} \otimes \mathbb{D}).$$

Let $\mathbb{P}_{\pm,\pm}$ be the corresponding projection operators of $L^2(\mathbb{T}\otimes\mathbb{T})$ onto $H^2_{\pm,\pm}(\mathbb{D}\otimes\mathbb{D})$.

The Hankel operators we study are given by

$$h_\varphi : H^2_{+,+}(\mathbb{D} \otimes \mathbb{D}) \to H^2_{-,-}(\mathbb{D} \otimes \mathbb{D})$$
$$h_\varphi f = \mathbb{P}_{-,-} M_\varphi f,$$

where M_φ denotes the operator of pointwise multiplication by φ initially taken to be in $L^2(\mathbb{T} \otimes \mathbb{T})$. One can see that these operators are “little” in the sense that they project onto the smallest possible subspace of L^2. Ferguson and Sadosky [9] examined so called *big* Hankel operators projecting onto the subspace of L^2 which is antianalytic in at least one variable and their relationship with a proper subset of product BMO.

The following theorem extends Nehari’s Theorem [16] to two complex variables and is basically equivalent to a result of S. Ferguson and M. Lacey [8] which we discuss below.

3.1. Theorem. *(Lacey, Terwilleger, Wick [15]) The Hankel operator h_φ is bounded iff there is a function $\psi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$ for which $\mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi$, and we have the equivalence*

$$\|h_\varphi\| \approx \inf\{\|\psi\|_\infty : \mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi\} \tag{3.1}$$
$$\approx \|\mathbb{P}_{-,-}\varphi\|_{BMO(\mathbb{T}\otimes\mathbb{T})}. \tag{3.2}$$

The above result, along with the ones below, have equivalent formulations in terms of Hankel operators on $H^2(\mathbb{C}_+ \otimes \mathbb{C}_+)$. In addition, this result can be phrased in the language of commutators. For a function $\varphi \in BMO(\mathbb{R}\otimes\mathbb{R})$ define the iterated commutator

$$C_\varphi := [[M_\varphi, H_1], H_2] \tag{3.3}$$

where H_j denotes the Hilbert transform computed in the coordinate j. In the next section, we show precisely how the commutator is related to Hankel operators.

In 1976, Coifman, Rochberg, and Weiss [4] showed in the case of one parameter that $\|[M_\varphi, H]\|_{2\to 2} \simeq \|\varphi\|_{BMO(\mathbb{R}^n)}$. From this they deduced the “weak factorization” result for functions in $H^1(\mathbb{R}^n)$ mentioned earlier. Ferguson and Lacey [8] obtained the two parameter result

$$\|C_\varphi\|_{L^2\to L^2} \simeq \|\varphi\|_{BMO(\mathbb{R}\otimes\mathbb{R})}.$$

Their paper is phrased on the real line, as the technical arguments are easier in this case. The difficulty in the two parameter setting is in proving the lower bound. To do so, one assumes that the function φ has BMO norm one, but has arbitrarily small $BMO(rec)$ norm. The key is to then use a covering lemma of Journé which shows that the $BMO(rec)$ norm can dominate the BMO norm if one puts a factor on the wavelet coefficients which depends on the rectangle associated to the wavelet and how embedded it is in the set.

The result of Ferguson and Lacey has two important of consequences. First, it gives another intrinsic characterization of product BMO in terms of commutators. Second, they obtain a weak factorization result in the product domain. This follows since for functions $f, g \in L^2(\mathbb{R}^2)$,

$$\langle C_\varphi f, g\rangle \approx \langle \varphi, \overline{\mathbb{P}_{-,-} f}\mathbb{P}_{+,+} g\rangle.$$

(This equivalence will be more apparent if one looks ahead to the proof of Corollary 4.3.) Thus the operator norm of C_φ is comparable to

$$\sup |\langle \varphi, fg\rangle| \simeq \|C_\varphi\|_{L^2\to L^2},$$

where the supremum is over functions f and g in $H^2(\mathbb{R}\otimes\mathbb{R})$ of norm one. But since $\varphi \in BMO(\mathbb{R}\otimes\mathbb{R})$ which is dual to $H^1(\mathbb{R}\otimes\mathbb{R})$, the Lacey-Ferguson result implies that the above norms are equivalent to

$$\|\varphi\|_{BMO(\mathbb{R}\otimes\mathbb{R})} = \sup |\langle \varphi, h\rangle|,$$

where the supremum is over functions h is in the unit ball of $H^1(\mathbb{R}\otimes\mathbb{R})$. Therefore, if we let $H^2(\mathbb{D}\otimes\mathbb{D})\widehat{\otimes}H^2(\mathbb{D}\otimes\mathbb{D}) \subset L^1(\mathbb{T}\otimes\mathbb{T})$ denote the injective tensor product with norm

$$\|h\|_{H^2\widehat{\otimes}H^2} := \inf\Big\{\sum_j \|f_j\|_{H^2}\|g_j\|_{H^2} \,:\, h = \sum_j f_j g_j\Big\},$$

we have

$$H^2(\mathbb{D}\otimes\mathbb{D})\widehat{\otimes}H^2(\mathbb{D}\otimes\mathbb{D}) = H^1(\mathbb{D}\otimes\mathbb{D}). \tag{3.4}$$

Using this equality, one is able to adopt the classical proof of Nehari to prove Theorem 3.1. One initially defines a linear functional on $H^2(\mathbb{D}\otimes\mathbb{D})\widehat{\otimes}H^2(\mathbb{D}\otimes\mathbb{D})$ which extends to one on $H^1(\mathbb{D}\otimes\mathbb{D})$ by (3.4). Also essential is the duality of $H^1(\mathbb{D}\otimes\mathbb{D})$ and $BMO(\mathbb{D}\otimes\mathbb{D})$. See Lacey, Terwilleger, and Wick [15] for details.

The weak factorization result of Ferguson and Lacey [8] is crucial to this proof as it is really the only place in the proof where two parameters, as opposed to one, make a difference. As mentioned earlier, in one dimension, we have a factorization of an H^1 function into a product of H^2 functions with equality of norms. Thus one attains equality in (3.1). A natural question to ask next is whether these results extend to the n dimensional polydisc. Lacey and Terwilleger [14] answered this in the affirmative. The proof of Ferguson and Lacey depends on the geometrical Journé Lemma which only holds in two dimensions in the form used. Lacey and Terwilleger were able to prove an n dimensional version of Theorem 3.1 by defining a range of "new" BMO spaces and using an inductive approach along with a nontrivial higher dimensional extension of Journé's lemma.

4. Compactness of the Hankel Operators on $H^2(\mathbb{D}\otimes\mathbb{D})$

Once establishing criteria for the boundedness of little Hankel operators of two complex variables, it is only natural to ask for necessary and sufficient conditions for the compactness of the Hankel operators. We have the following refinement of Theorem 3.1.

4.1. Theorem. *(Lacey, Terwilleger, Wick [15]) h_φ is compact iff $\mathbb{P}_{-,-}\varphi$ is in the closure of $C(\mathbb{T}\otimes\mathbb{T})$ with respect to the BMO topology.*

We call this last space $VMO(\mathbb{D}\otimes\mathbb{D})$ in view of the classical result of Sarason [19] in one parameter. This space has an equivalent characterization in terms of wavelets and Carleson measures which we take up in Section 6.

To establish the one parameter version of Theorem 4.1, one uses the Hartman compactness criteria along with Sarason's representation of VMO functions in terms of continuous functions. As noted earlier, we do not have such a representation in two parameters. However, we do obtain a version of Hartman's theorem which will be discussed in Section 5. To prove Theorem 4.1, one can take a direct approach using Theorem 3.1 along with the following lemma which is also used in the next section. Let S_j be the shift operator on $H^2_{+,+}(\mathbb{D}\otimes\mathbb{D})$ associated with multiplication by z_j, for $j=1,2$.

4.2. Lemma. *(Lacey, Terwilleger, Wick [15]) For all compact operators $K : H^2_{+,+}(\mathbb{D}\otimes\mathbb{D}) \longrightarrow H^2_{-,-}(\mathbb{D}\otimes\mathbb{D})$, as $n,m\to\infty$ we have $\|KS_j^nS_{j'}^m\|\longrightarrow 0$, for $j,j'=1,2$.*

Proof. If $j=j'$, we simply have S_j^{n+m}, which is one of the multiplication operators, and the argument we give will also work. Thus by symmetry we can suppose that $j=1$ and $j'=2$. Since we can approximate any compact operator by finite rank operators, it is also enough to deal with rank one operators. Furthermore, we actually check the claim on a dense class of rank one operators. In particular, take K to be defined by

$$K(f)=\langle f,g\rangle h \qquad \forall f\in H^2_{+,+}(\mathbb{D}\otimes\mathbb{D})$$

with $h\in H^2_{-,-}(\mathbb{D}\otimes\mathbb{D})$ and $g\in H^2_{+,+}(\mathbb{D}\otimes\mathbb{D})$ a polynomial of degree strictly less than n in the z_1 variable and strictly less than m in the z_2 variable. For a polynomial $p(z_1,z_2)=\sum_{i=0}^r\sum_{k=0}^s a_{ik}z_1^iz_2^k$ in $H^2_{+,+}(\mathbb{D}\otimes\mathbb{D})$,

$$S_1S_2p(z_1,z_2)=\sum_{i=0}^r\sum_{k=0}^s a_{ik}z_1^{i+1}z_2^{k+1}.$$

The adjoint operator is then a backward shift given by

$$S_1^*S_2^*p(z_1,z_2)=\sum_{i=0}^{r-1}\sum_{k=0}^{s-1} a_{(i+1)(k+1)}z_1^iz_2^k.$$

Therefore $(S_1^*)^n(S_2^*)^mg=\mathbb{P}_{+,+}(z_1^{-n}z_2^{-m}g)=0$, which means $KS_1^nS_2^m=0$. □

If we assume h_φ is compact, then we use Lemma 4.2 with the shift operators S_1^n, S_2^n, and $S_1^nS_2^n$ to obtain that the operator norm of a composition of a Hankel operator with the shift operator is arbitrarily small. But we can think of these as Hankel operators where the frequencies of the Fourier coefficients are shifted by $-n$. Thus by Theorem 3.1, the result of Theorem 4.1 is deduced. Again see the paper of Lacey, Terwilleger, and Wick [15] for details.

The compactness result for Hankel operators implies compactness of the commutator. In the commutator setting, we need the real Hardy, BMO, and VMO spaces. Let us define the real VMO space

$$VMO(\mathbb{R}\otimes\mathbb{R}):=\operatorname{clos}_{BMO}C_0^\infty(\mathbb{R}\otimes\mathbb{R})$$

where C_0^∞ denotes the space of smooth, compactly supported functions.

4.3. Corollary. *(Lacey, Terwilleger, Wick [15]) The commutator C_φ is compact iff $\varphi \in VMO(\mathbb{R}\otimes\mathbb{R})$.*

Proof. This proof uses the characterization of the compactness of the Hankel operators that we have already given in Theorem 4.1. While we discussed that proof on the bi-torus, it has an equivalent formulation on $\mathbb{R}\otimes\mathbb{R}$. Therefore we show how to identify the nested commutators C_φ with the little Hankel operators when working in $\mathbb{R}\otimes\mathbb{R}$.

The first observation is that the Hilbert transforms can be written in terms of the projection operators:

$$H_1 = \mathbb{P}_{+,+} + \mathbb{P}_{+,-} - \mathbb{P}_{-,+} - \mathbb{P}_{-,-} \text{ and } H_2 = \mathbb{P}_{+,+} + \mathbb{P}_{-,+} - \mathbb{P}_{+,-} - \mathbb{P}_{-,-}.$$

Therefore the commutator $C_\varphi := [[M_\varphi, H_1],\, H_2]$ can be written

$$C_\varphi = 4(\mathbb{P}_{+,+}M_\varphi\mathbb{P}_{-,-} - \mathbb{P}_{+,-}M_\varphi\mathbb{P}_{-,+} - \mathbb{P}_{-,+}M_\varphi\mathbb{P}_{+,-} + \mathbb{P}_{-,-}M_\varphi\mathbb{P}_{+,+}).$$

The commutator acts on $L^2(\mathbb{R}\otimes\mathbb{R}) = \oplus_{\varepsilon\in\{\pm,\pm\}} H^2_\varepsilon(\mathbb{D}\otimes\mathbb{D})$, and so this gives four relevant Hankel operators as maps from $H^2_\varepsilon(\mathbb{R}\otimes\mathbb{R}) \longrightarrow H^2_{-\varepsilon}(\mathbb{R}\otimes\mathbb{R})$, where $\varepsilon \in \{\pm,\pm\}$ and $-\varepsilon$ is conjugate to ε. They are defined

$$h_{\varphi,\varepsilon} = \mathbb{P}_{-\varepsilon}M_\varphi \,:\, H^2_\varepsilon(\mathbb{R}\otimes\mathbb{R}) \longrightarrow H^2_{-\varepsilon}(\mathbb{R}\otimes\mathbb{R}).$$

By a similar proof as that for Theorem 4.1, we have the fact that any of these operators is compact iff $\mathbb{P}_{-\varepsilon}\,\varphi \in VMO(\mathbb{R}\otimes\mathbb{R})$. But $\varphi = \oplus_{\varepsilon\in\{\pm,\pm\}}\mathbb{P}_{-\varepsilon}\,\varphi$. Thus C_φ is compact iff each of the $h_{\varphi,\varepsilon}$ are compact iff $\varphi \in VMO(\mathbb{R}\otimes\mathbb{R})$. □

5. Hartman's Compactness Criteria on the Bi-disc

We have already discussed one compactness criterion for little Hankel operators. If the symbol of the Hankel operator is assumed to be bounded, then there is a different viewpoint in terms of the essential norm and Hartman's Compactness Criteria. The essential norm is defined to be

$$\|h_\varphi\|_e := \inf\{\|h_\varphi - K\| \,:\, K\colon H^2_{+,+}(\mathbb{T}\otimes\mathbb{T}) \to H^2_{-,-}(\mathbb{T}\otimes\mathbb{T}) \text{ is compact}\}.$$

Observe that $\|h_\varphi\|_e = 0$ iff h_φ is compact. The key step is the following theorem which gives us a way to quantify the essential norm of Hankel operators in terms of the distance from a certain space of functions. Let $\mathcal{L}^p_{\pm,\pm}(\mathbb{T}\otimes\mathbb{T})$ be the space of functions $b \in L^p(\mathbb{T}\otimes\mathbb{T})$ such that $\mathbb{P}_{\pm,\pm}b = 0$. One should be careful to note the difference of the meaning of the subscripts on these spaces as opposed to $H^p_{\pm,\pm}(\mathbb{T}\otimes\mathbb{T})$, where, for example, if $b \in H^2_{-,-}(\mathbb{T}\otimes\mathbb{T})$, then $\mathbb{P}_{-,-}b \neq 0$ while the projections onto the other orthants are zero. We will principally be concerned with the space $\mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$ which takes the place of bounded analytic functions in one parameter. It is not enough to consider $H^\infty_{+,+}(\mathbb{T}\otimes\mathbb{T})$, as this space does not contain all the functions whose anti-analytic projection in each variable is zero. Recall that $\mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T}) + C(\mathbb{T}\otimes\mathbb{T})$ is the space of functions $\varphi \in L^\infty(\mathbb{T}\otimes\mathbb{T})$ that have a decomposition of the form $\psi + g$ with $\psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$ and $g \in C(\mathbb{T}\otimes\mathbb{T})$.

5.1. Theorem. *(Lacey, Terwilleger, Wick [15]) Let $\varphi \in L^\infty(\mathbb{T}\otimes\mathbb{T})$. Then*

$$\|h_\varphi\|_e \approx \operatorname{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-} + C).$$

Assuming this theorem, we can easily deduce a two parameter version of Hartman's Theorem.

5.2. THEOREM. *(Lacey, Terwilleger, Wick [15]) Let $\varphi \in L^\infty(\mathbb{T}\otimes\mathbb{T})$. Then the following are equivalent*

(i) h_φ *is compact.*

(ii) $\varphi \in \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T}) + C(\mathbb{T}\otimes\mathbb{T})$.

(iii) *there exists a* $g \in C(\mathbb{T}\otimes\mathbb{T})$ *such that* $h_\varphi = h_g$.

It should be noted that in light of Theorem 4.1, the condition that φ be bounded is not needed to prove (i) and (iii) are equivalent.

The proof of Theorem 5.1 found in [15] follows the presentation of Hartman's Theorem in V. Peller's book [17]. Thus the structure of the proof is very similar to the one parameter version. However, at one point, a key fact will not be available to us since we are working with $\mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$ rather than $H^\infty_{+,+}(\mathbb{T}\otimes\mathbb{T})$. It turns out we can get around this with a suitable characterization of the space $\mathcal{L}^\infty_{-,-} + C$. In the spirit of the one-variable case we have the following theorem.

5.3. THEOREM. *(Lacey, Terwilleger, Wick [15]) $\mathcal{L}^\infty_{-,-} + C$ is a closed subspace of $L^\infty(\mathbb{T}\otimes\mathbb{T})$, and moreover*

$$\mathcal{L}^\infty_{-,-} + C = \mathrm{clos}_{L^\infty}\left(\bigcup_{n,m=0}^{\infty} \overline{z}_1^n \overline{z}_2^m \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})\right).$$

This is a slightly weaker result than what one would find in one complex variable. The one variable analog of the space above, $H^\infty + C$, is in fact a sub-algebra of $L^\infty(\mathbb{T})$. In higher dimensions, $\mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$ is not closed under multiplication as $H^\infty_{+,+}(\mathbb{D}\otimes\mathbb{D})$ is, so $\mathcal{L}^\infty_{-,-} + C$ will not be a sub-algebra. However, the formulation above is enough for our purposes. To prove this theorem, one uses that for $\varphi \in C(\mathbb{T}\otimes\mathbb{T})$

$$\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}) = \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-} \cap C).$$

One last fact is crucial for the proof of Theorem 5.1. In one parameter, the norm of a Hankel operator is equal to the distance in L^∞ of its symbol from $H^\infty(\mathbb{D})$. The following is the natural extension to the bi-disc of this fact.

5.4. LEMMA. *(Lacey, Terwilleger, Wick [15]) Let $\varphi \in L^\infty(\mathbb{T}\otimes\mathbb{T})$. Then*

$$\|h_\varphi\| \approx \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}) := \inf\{\|\varphi-\psi\|_\infty : \psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})\}.$$

PROOF. Clearly if $\varphi \in L^\infty(\mathbb{T}\otimes\mathbb{T})$ and $\psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$, then given $f \in H^2_{+,+}(\mathbb{D}\otimes\mathbb{D})$, we have $\psi f \in \mathcal{L}^2_{-,-}(\mathbb{T}\otimes\mathbb{T})$. Therefore,

$$h_{\varphi-\psi} f = \mathbb{P}_{-,-}(\varphi-\psi) f = \mathbb{P}_{-,-}\varphi f - \mathbb{P}_{-,-}\psi f = \mathbb{P}_{-,-}\varphi f = h_\varphi f,$$

which implies

$$\|h_\varphi\| = \|h_{\varphi-\psi}\|.$$

On the other hand, Nehari's Theorem on the bi-disc, Theorem 3.1, says that there exists a function $\eta \in L^\infty$ such that

$$\|h_{\varphi-\psi}\| \approx \inf\{\|\eta\|_\infty : \mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\eta\}.$$

Since $\varphi - \psi \in L^\infty$, this implies that

$$\inf\{\|\varphi-\psi\|_\infty : \psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})\} \approx \|h_{\varphi-\psi}\| = \|h_\varphi\|.$$

Thus $\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}) \approx \|h_\varphi\|$. □

We now sketch a proof of Theorem 5.1. We reference [15] for details. For a compact operator K as in the definition of the essential norm, we apply $h_\varphi - K$ to the shift operators $S_1^n S_2^m$ and estimate the operator norm which is at most $\|h_\varphi - K\|$. By Lemma 4.2, $\|KS_1^n S_2^m\|$ tends to zero as $n, m \to \infty$. The other term $\|h_\varphi S_1^n S_2^m\| = \|h_{z_1^n z_2^m \varphi}\|$ can be bounded below by $\operatorname{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-} + C)$ using Lemma 5.4 and Theorem 5.3. In the other direction, one uses that for $g \in C(\mathbb{T}\otimes\mathbb{T})$, the operator h_g is compact. We then obtain an upper bound on the essential norm by taking $\inf_{g\in C}\|h_\varphi - h_g\|$ which is no more than $\operatorname{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-} + C)$ by Lemma 5.4.

6. *VMO* and Carleson Measures

The final theorem we prove is a duality result which gives the counterpart of the well-known $H^1 - BMO$ duality in the product setting.

6.1. Theorem. *(Lacey, Terwilleger, Wick [15]) We have* $VMO(\mathbb{R}\otimes\mathbb{R})^* = H^1(\mathbb{R}\otimes\mathbb{R})$.

In order to prove the theorem above, we state an equivalent form of the definition of $VMO(\mathbb{R}\otimes\mathbb{R})$ in terms of Carleson measures and, in particular, in a variant of the definition of Chang and Fefferman. Recall

$$f \in BMO \iff \|f\|_{BMO(\mathbb{R}\otimes\mathbb{R})} \approx \sup_U \Big[|U|^{-1} \sum_{R\subset U} |\langle f, w_R\rangle|^2\Big]^{1/2} < \infty. \tag{6.1}$$

Strictly speaking (6.1) is not a classical Carleson measure, but for a function in BMO we can associate a Carleson measure in a similar fashion to the one parameter setting. This associated measure looks very similar to what appears in (6.1), and when dealing with VMO we can similarly associate a vanishing Carleson measure.

For a fixed choice of wavelet w, set $FW(w)$ to be the linear space of all finite linear combinations of wavelets $\{w_R : R \in \mathcal{D}\times\mathcal{D}\}$.

6.2. Proposition. *(Lacey, Terwilleger, Wick [15]) For a fixed choice of wavelet w, the following are equivalent.*

(i) *A function b is in* $VMO(\mathbb{R}\otimes\mathbb{R}) := \operatorname{clos}_{BMO} C_0^\infty(\mathbb{R}\otimes\mathbb{R})$.
(ii) *b is in the closure, in BMO norm, of $FW(w)$.*
(iii) *$b \in BMO(\mathbb{R}\otimes\mathbb{R})$, and writing $R = R_1 \times R_2$ for a rectangle R,*

$$\lim_{N\to\infty}\Big\| \sum_{\substack{R\in\mathcal{D}\times\mathcal{D}\\ |\log|R_1||+|\log|R_2||>N}} \langle b, w_R\rangle w_R\Big\|_{BMO(\mathbb{R}\otimes\mathbb{R})} = 0,$$

$$\lim_{N\to\infty}\Big\| \sum_{\substack{R\in\mathcal{D}\times\mathcal{D}\\ R\not\subset\{|x|<N\}}} \langle b, w_R\rangle w_R\Big\|_{BMO(\mathbb{R}\otimes\mathbb{R})} = 0.$$

These conditions are independent of the choice of wavelet basis.

First, in the following lemma we indicate why the conditions in this proposition are independent of the choice of wavelet.

6.3. Lemma. *(Lacey, Terwilleger, Wick [15]) For any two choices of w and w',*

$$\operatorname{clos}_{BMO}\mathrm{FW}(w) = \operatorname{clos}_{BMO}\mathrm{FW}(w').$$

Proof. First observe that both spaces are invariant under dilations by factors of 2 since the wavelets are adapted to dyadic rectangles. Recall that the wavelets w and w' are products of functions of one variable which are bounded, piecewise continuous, rapidly decreasing, and have mean value zero. Thus we have

$$\sum_{R\in\mathcal{D}\times\mathcal{D}} |\langle w', w_R\rangle| < \infty.$$

This fact clearly implies that w', and similarily each wavelet w'_R, are contained in $\mathrm{clos}_{BMO}\mathrm{FW}(w)$. Hence the same is true of each element of $\mathrm{FW}(w')$. □

Proof of Proposition 6.2. Lemma 6.3 allows us to make particular choices for w in different parts of our proof. In addition, we suppress the explicit choice of wavelet in our notation. It is a routine calculation to verify that (ii) is equivalent to (iii). The first equation in (iii) sums over rectangles which have arbitrarily large or small scale, while in the second equation, the sum is over rectangles which stay away from some large ball centered at the origin. Clearly $b \in FW$ if and only if the sums equal zero for some N large enough. In the limiting case, $b \in \mathrm{clos}_{BMO}\mathrm{FW}$ if and only if the BMO norms in (iii) tend to zero.

The case (ii) implies (i) is also rather simple in light of Lemma 6.3. Choose the wavelet to be smooth and have compact spatial support, in which case it is clear that $FW \subset C_0^\infty$. Therefore

$$\mathrm{clos}_{BMO}FW \subseteq \mathrm{clos}_{BMO}C_0^\infty.$$

It remains to show that $b \in VMO$ implies (iii), which completes the string of equivalences. This time we only need the wavelet to be a Schwartz function. Thus we have the decay estimate

$$|w_R(x)| \le C_n \frac{1}{|R|^{1/2}}\left(1+\frac{|x-c(R)|}{|R|}\right)^{-n}, \tag{6.2}$$

where n is an arbitrarily large integer. To verify that a function in C_0^∞ satisfies condition (iii) of the proposition, one uses the estimates below, valid for all $f \in C_0^\infty$ with constants that depend upon the choice of f and n in (6.2).

$$|\langle f, w_R\rangle| \lesssim \begin{cases} |R|^{3/2}, & |R_1|, |R_2| < 1 \\ \frac{|R_1|}{\sqrt{|R_2|}}, & |R_1| < 1 < |R_2| \\ |R|^{-1/2}, & |R_1|, |R_2| > 1. \end{cases}$$

If $|R_1|, |R_2| < 1$, then one can pull the L^∞ norm of f out of the inner product and integrate $|w_R|$ using the bound in (6.2). On the other hand, if $|R_1|, |R_2| > 1$, one can bound the wavelet by $|R|^{-1/2}$ and integrate $|f|$. Thus a function in C_0^∞ can be well approximated in BMO norm by finite sums of wavelets. □

Proof of Theorem 6.1. The inclusion $H^1 \subset VMO^*$ follows from H^1–BMO duality and the fact that $VMO \subset BMO$. Indeed, $\|f\|_{H^1} \ge |\langle b, f\rangle|$ for all $b \in VMO$, and so $\|f\|_{H^1} \ge \|f\|_{VMO^*}$. To show the reverse containment we use the result $VMO = \mathrm{clos}_{BMO}FW$ from Proposition 6.2. Given $f, b \in FW(w)$, there exist finite collections $\mathcal{S}, \mathcal{T} \subset \mathcal{D}\times\mathcal{D}$ and coefficients a_R and c_R such that

$$f = \sum_{R\in\mathcal{S}} a_R w_R \quad \text{and} \quad b = \sum_{R\in\mathcal{T}} c_R w_R.$$

By orthogonality of the wavelet basis, $|\langle f, b\rangle| = |\sum_{R\in\mathcal{S}\cap\mathcal{T}} a_R c_R|$. So we can choose b with wavelet coefficients so that

$$\Big|\sum_{R\in\mathcal{S}\cap\mathcal{T}} a_R c_R\Big| \approx \sum_{R\in\mathcal{S}\cap\mathcal{T}} |a_R| \approx \|f\|_{H^1}.$$

Therefore

$$\|f\|_{VMO^*} = \sup_{\substack{b\in VMO\\ \|b\|_{BMO}=1}} |\langle f, b\rangle| \geq C\|f\|_{H^1},$$

for $f \in FW(w)$. Thus we need to show that $\operatorname{clos}_{H^1} FW = H^1$. Gundy and Stein [11] established a maximal function characterization of product H^1 which will imply the Littlewood-Paley inequalities

$$\|f\|_{H^1} \approx \Big\|\Big[\sum_{R\in\mathcal{D}\times\mathcal{D}} \frac{|\langle f, w_R\rangle|^2}{|R|}\mathbf{1}_R\Big]^{1/2}\Big\|_1.$$

This is also related to the atomic decomposition of H^1 given by Chang and Fefferman [2,3]. The above equivalence shows that $\operatorname{clos}_{H^1} FW = H^1$ and completes the proof. □

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