Energy Partition on Fractals

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ABSTRACT. The energy of a function defined on a post–critically finite self–similar fractal can be written as a sum of directional energies. We show, under mild hypotheses, that each directional energy is a fixed multiple of the total energy, and we compute the multiple for a one–parameter family of energy forms on the Sierpinski gasket. For the standard one, the result is an equipartition of energy principle. Also we discuss the energy partition for general p.c.f. fractals, and the relation of it to the uniqueness and stability of a self-similar Dirichlet form.

1. INTRODUCTION

The energy of a function u on an open set Ω in $\mathbb{R}^n$, defined to be a multiple (taken to be 1 for simplicity) of the integral of $|\nabla u|^2$, can be written as a sum of directional energies $\int_\Omega |\partial u/\partial x_j|^2 dx$. There are no required relations between the directional energies in this case, as may be seen just taking u to be a linear function. But something quite different happens on fractals.

Consider the example of the familiar Sierpinski gasket SG, the attractor of the iterated function system (IFS) consisting of the 3 contractions $F_j(x) = \frac{1}{2}(x + v_j)$, where v_j are the vertices of an equilateral triangle in the plane. Define graphs Γ_m with vertices V_m inductively as follows: Γ_0 is the complete graph on $V_0 = \{v_0, v_1, v_2\}$, and $V_m = \bigcup_{i=0,1,2} F_i V_{m-1}$ with $x \sim_m y$ if $x = F_w v_j$, $y = F_w v_k$ for some word $w = (w_1, \dots, w_m)$ of length m, where $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ (see Figure 1).

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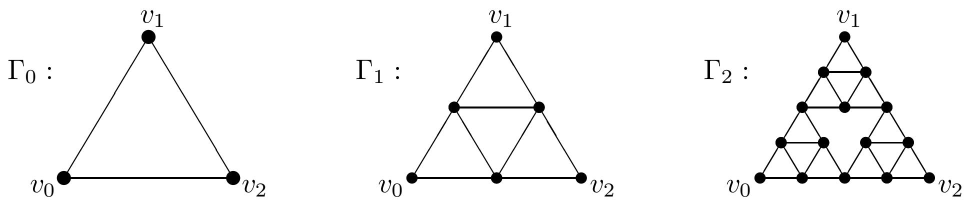

FIGURE 1. The first 3 graphs Γ_0, Γ_1, and Γ_2 approximating SG.

Then SG is the closure of $\bigcup_m V_m$, and we may define the standard energy (or Dirichlet form) as

$$\mathcal{E}(u,u) = \lim_{m\to\infty} \mathcal{E}_m(u,u) \tag{1.1}$$

where

$$\mathcal{E}_m(u,u) = \left(\frac{5}{3}\right)^m \sum_{x\sim_m y} (u(x) - u(y))^2. \tag{1.2}$$

The sequence $\{\mathcal{E}_m(u,u)\}$ is monotone increasing, and the domain of $\mathcal{E}$, the space of functions of finite energy, is defined to be the space of functions u for which the limit (1.1) is finite [Ki1,Ki2]. (It is important to observe that points have positive capacity for this Dirichlet form, so that functions in dom $\mathcal{E}$ must be continuous, and the pointwise values in (1.2) are well–defined.) We may also write the approximate energy $\mathcal{E}_m$, using cyclic notation for subscripts of v_i, as

$$\mathcal{E}_m(u,u) = \sum_{i=0}^{2} \mathcal{E}_m^{(i)}(u,u) \tag{1.3}$$

for

$$\mathcal{E}_m^{(i)}(u,u) = \left(\frac{5}{3}\right)^m \sum_{|w|=m} (u(F_w v_{i+1}) - u(F_w v_i))^2, \tag{1.4}$$

partitioning the sum along the 3 directions of the edges. These directional approximate energies are no longer monotone increasing, and one might wonder whether or not the limits

$$\mathcal{E}^{(i)}(u,u) = \lim_{m\to\infty} \mathcal{E}_m^{(i)}(u,u) \tag{1.5}$$

defining directional energies exist. In Section 2 we show that the limits exist for any function of finite energy, and indeed $\mathcal{E}^{(i)}(u,u) = \frac{1}{3}\mathcal{E}(u,u)$, so we have equipartition for this symmetric energy form. In fact, for harmonic functions we have the more precise result

$$\mathcal{E}_m^{(i)}(u,u) - \frac{1}{3}\mathcal{E}(u,u) = \left(\frac{4}{5}\right)^m \left(\mathcal{E}_0^{(i)}(u,u) - \frac{1}{3}\mathcal{E}(u,u)\right) \tag{1.6}$$

(in this case $\mathcal{E}(u,u) = \mathcal{E}_m(u,u)$ for all m). Once we have equipartition for harmonic functions, it follows immediately for piecewise harmonic splines, and then for all functions of finite energy because harmonic splines are dense in energy norm ([Ki2] Lemma 3.2.17). However, (1.6) does not hold for a wider class of functions.

In Section 3 we consider analogous results for a one–parameter family of self–similar Dirichlet forms on SG, involving a more complicated analog of (1.2) for the approximate energy. The approximate energy again splits via (1.3) into three directional energies via the analog of (1.4), with the limits (1.5) existing and yielding a partition of energy

$$\mathcal{E}^{(i)}(u,u) = a_i\mathcal{E}(u,u) \tag{1.7}$$

where the coefficients a_i are given by explicit algebraic functions of the parameter. One important observation arising from the details of this family of examples is that there is apparently no simple recipe for finding the coefficients a_i in (1.17).

In this case we do not have an exact equality like (1.6), but only an estimate that implies exponential convergence. It should be noted already in (1.6) that the convergence ratio $4/5$ is quite close to 1, so numerical values of the approximate directional energies $\mathcal{E}_m^{(i)}(u,u)$ will not reveal the energy partition with much accuracy in the range of values of m for which computations are feasible. Nevertheless, we must confess that the discovery of the phenomenon of energy partition arose from the examination of the results of numerical experiments which revealed (1.6).

In Section 4 we show that the analogs of (1.5) and (1.7) hold for self–similar Dirichlet forms on general post–critically finite (p.c.f.) self–similar fractals [Ki1, Ki2], under very mild assumptions. In the general case we do not have an explicit expression for the coefficients in (1.7). Judging by the explicit expression for the examples considered in Section 3, any explicit expression would have to be quite complicated.

The results in this paper are closely connected to works on self-similar Dirichlet forms by Metz [M1,M2], R. Peirone [P] and Sabot [Sa] (see also references in these papers). The article [KK] deals with a probabilistic version of these questions. If a self-similar Dirichlet form exists and a certain irreducibility condition is satisfied, our results imply its uniqueness and stability under discrete approximations (another way of dealing with this problem can be found in [P]). In comparison to the previous results, our presentation, based on the standard Perron-Frobenius theory, is shorter and simpler, and also deals with some situations not previously covered (see Remark 4.8 on "essential fixed points"). However one should note that the most important part of [M1,M2,Sa] concerns existence and uniqueness problem for self–similar Dirichlet forms, while in our paper the existence is assumed. In particular, the main object of study in [M1,M2,Sa] is a nonlinear renormalization map, whereas in our paper we significantly simplify the situation by considering the linearization of this map, namely its derivative at a fixed point.

The partition of energy by directions should be contrasted with the distribution of energy by location. Indeed, there exist energy measures ν_u such that

$$\mathcal{E}(u,u) = \int_K d\nu_u \tag{1.8}$$

where K denotes the whole fractal, and $\nu_u(A)$ represents the energy localized to the set A. A straightforward consequence of our results is that we also have a partition of the energy measure, $\nu_u = \sum \nu_u^{(i)}$ and $\nu_u^{(i)} = a_i\nu_u$, where $\nu_u^{(i)}$ are directional energy measures. But for most fractals ν_u is singular with respect to the standard Hausdorff measure on K (see [Ku] or [BST]). We can paraphrase the results as follows: *energy distribution is geographically wild but directionally tame*.

One consequence of the energy partition is the observation that the analog of elliptic pde in divergence form on these fractals resembles more the Sturm–Liouville ode's on an interval. In other words, the operator

$$Lu(x) = \sum_j \sum_k \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right) \tag{1.9}$$

in $\mathbb{R}^n$ is associated to the Dirichlet form

$$\mathcal{E}(u,u) = \sum_j \sum_k \int_\Omega a_{jk}(x)\frac{\partial u}{\partial x_j}\,\frac{\partial u}{\partial x_k}dx, \tag{1.10}$$

which seems to suggest that one consider

$$\lim_{m\to\infty}\left(\frac{5}{3}\right)^m \sum_{i=0}^{2}\sum_{|w|=m}\frac{1}{2}(a_i(F_w v_{i+1}) + a_i(F_w v_{i-1})) \times (u(F_w v_{i+1}) - u(F_w v_{i-1}))^2 \tag{1.11}$$

as the analogous non–self–similar Dirichlet form on SG, for a vector $(a_0(x), a_1(x), a_2(x))$ of continuous functions. But this is just equal to

$$\lim_{m\to\infty}\left(\frac{5}{3}\right)^m \sum_{x\sim_m y}\frac{1}{2}(a(x)+a(y))(u(x)-u(y))^2 \tag{1.12}$$

for $a(x) = \frac{1}{3}(a_0(x) + a_1(x) + a_2(x))$, which is the analog of

$$\int_0^1 a(x)u'(x)^2 dx \tag{1.13}$$

on the interval, associated with the Sturm–Liouville operator $(au')'$. For some preliminary work on this see the web site http://www.mathlab.cornell.edu/~jstanley/

2. Equipartition on SG

We consider the standard self–similar Dirichlet form on SG defined by (1.1) and (1.2). A continuous function u on SG is called *harmonic* if it minimizes energy for given boundary values $u|_{V_0}$. Harmonic functions have the property that $\mathcal{E}_m(u,u)$ is independent of m. They are determined by their boundary values by a local linear extension algorithm

$$u(F_w F_i v_{i+1}) = \frac{2}{5}u(F_w v_i) + \frac{2}{5}u(F_w v_{i+1}) + \frac{1}{5}u(F_w v_{i-1}). \tag{2.1}$$

We can also write this in matrix form

$$\begin{pmatrix} u(F_w F_i v_0) \\ u(F_w F_i v_1) \\ u(F_w F_i v_2) \end{pmatrix} = A_i \begin{pmatrix} u(F_w v_0) \\ u(F_w v_1) \\ u(F_w v_2) \end{pmatrix} \tag{2.2}$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

The space of harmonic functions is only 3 dimensional, but by creating piecewise harmonic splines we can approximate any function of finite energy, so most questions about energy can be reduced to the corresponding questions for harmonic functions.

Lemma 2.1. *For any harmonic function,* (1.6) *holds.*

Proof. Denote by $d_m^{(i)}$ the discrepancy

$$\mathcal{E}_m^{(i)}(u,u) - \frac{1}{3}\mathcal{E}_m(u,u). \tag{2.4}$$

It suffices to show

$$d_{m+1}^{(i)} = \frac{4}{5}d_m^{(i)}. \tag{2.5}$$

Consider the case $m = 0$. By a linear substitution and a rotation we may reduce to the case $u(v_0) = 0$, $u(v_1) = x$, $u(v_2) = 1$ with $0 \le x \le 1$. Then $u(F_0v_1) = (2x+1)/5$, $u(F_1v_2) = (2+2x)/5$, $u(F_2v_0) = (2+x)/5$ by (2.1). A routine computation yields (2.5). For example, $\mathcal{E}_0 = \mathcal{E}_1 = 2x^2 - 2x + 2$, $\mathcal{E}_0^{(2)} = 1$, $\mathcal{E}_1^{(2)} = \frac{2}{15}x^2 - \frac{2}{15}x + \frac{14}{15}$, so that $d_0^{(2)} = \frac{1}{3}(1 + 2x - 2x^2)$ and $d_1^{(2)} = \frac{4}{15}(1 + 2x - 2x^2)$, and the other directions are similar.

The general case is essentially the same argument. Because of (1.4) the computation of $d_m^{(i)}$ involves a sum over all words w of length m of a discrepancy over the cell F_wK where we use the $m = 0$ case for the harmonic function $u \circ F_w$. □

Theorem 2.2. *Equipartition of energy,* (1.7) *with* $a_0 = a_1 = a_2 = 1/3$, *holds for all functions of finite energy* (*in particular, the limit* (1.5) *exists*).

Remark 1. The passage from Lemma 2.1 to Theorem 2.2 is generic, and applies to the cases in the next sections.

Proof. This is obvious for harmonic functions from Lemma 2.1. Define a harmonic spline of order n to be a continuous function u such that $u \circ F_w$ is a harmonic function for all words w of length n. Note that $\mathcal{E}_m(u,u) = \mathcal{E}(u,u)$ for all $m \ge n$ for such functions, and from Lemma 2.1 it follows that (2.5) holds for all $m \ge n$. So (1.7) holds for harmonic splines. By Lemma 3.2.17 of [Ki2], the interpolating harmonic splines u_n approximate a general function u of finite energy:

$$\lim_{n\to\infty} \mathcal{E}(u - u_n, u - u_n) = 0. \tag{2.6}$$

Because all the terms in (1.3) are positive,

$$\mathcal{E}_m^{(i)}(u - u_n, u - u_n) \le \mathcal{E}_m(u - u_n, u - u_n) \le \mathcal{E}(u - u_n, u - u_n),$$

so (2.6) implies the existence of the limit (1.5) and

$$\lim_{n\to\infty} \mathcal{E}^{(i)}(u - u_n, u - u_n) = 0.$$

The equipartition of energy is then inherited from u_n to u. □

Remarks.

(1) The existence of the limit (1.5) for one value of i does not imply that the function has finite energy, however. For example, a function of the form $u(x,y) = g(y)$ would have horizontal energy equal to zero, but infinite energy for any nonconstant g.

(2) When u is harmonic we have a rate of convergence $O((\frac{4}{5})^m)$ for $\mathcal{E}_m^{(i)}(u,u) - \frac{1}{3}\mathcal{E}(u,u)$, but the theorem does not guarantee a rate of convergence under the hypothesis that u has finite energy. If we are willing to assume more

in the way of "smoothness" for u, then we can again obtain an exponential rate of convergence $O(\gamma^m)$ for some $\gamma < 1$ (however 4/5 is a lower bound for γ). For example, suppose u is in the domain of the Laplacian associated with $\mathcal{E}$ and the standard normalized Hausdorff measure μ. This means u is in the domain of $\mathcal{E}$ and there exists a continuous function Δu such that $\mathcal{E}(u,v) = -\int(\Delta u)v\,d\mu$ for all v in dom $\mathcal{E}$ vanishing on the boundary. Then Theorem 4.8 in [SU] gives the estimate

$$\mathcal{E}(u-u_n, u-u_n)^{1/2} \le c5^{-n/2}$$

(under the slightly weaker hypothesis that $\Delta u \in L^2$). By routine estimates this yields

$$\begin{aligned} \left|\mathcal{E}_m^{(i)}(u,u) - \frac{1}{3}\mathcal{E}(u,u)\right| & \\ &\le \left|\mathcal{E}_m^{(i)}(u_n,u_n) - \frac{1}{3}\mathcal{E}(u_n,u_n)\right| + c|\mathcal{E}(u,u) - \mathcal{E}(u_n,u_n)| \\ &\le \left|\mathcal{E}_m^{(i)}(u_n,u_n) - \frac{1}{3}\mathcal{E}(u_n,u_n)\right| + c\mathcal{E}(u,u)^{1/2}5^{-n/2} \end{aligned}$$

and Lemma 2.1 implies

$$\left|\mathcal{E}_m^{(i)}(u_n,u_n) - \frac{1}{3}\mathcal{E}(u_n,u_n)\right| \le c\left(\frac{4}{5}\right)^{m-n}$$

for $m \ge n$. (Here we use c to denote a generic constant.) If we choose $n = [\beta m]$ for $0 < \beta < 1$ we obtain a rate of convergence $O((\frac{4}{5})^{(1-\beta)m}) + O((\frac{1}{5})^{m\beta/2})$. The optimal choice of β makes these two rates equal, so

$$\beta = \frac{\log 5 - \log 4}{\frac{3}{2}\log 5 - \log 4} \approx .2170947 \text{ and } \gamma \approx .8397087.$$

We could reduce γ further by assuming that u is in the domain of suitable powers of Δ, getting down to $.8 + \varepsilon$ for any $\varepsilon > 0$, but it seems hardly worth the effort.

The simple proof of the Lemma does not reveal clearly what is going on, so we give a more elaborate explanation to set the stage for later examples. The key idea is to introduce the operator T defined in (2.11) below. Let Q denote a general quadratic form that annihilates constants over a 3 dimensional space that we will interpret to be the boundary values $[u] = (u(v_0), u(v_1), u(v_2))$ of a harmonic function. Q is represented by a 3×3 symmetric matrix with row sums zero. The space of such quadratic forms is 3–dimensional, and includes the energy form

$$Q_E = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{2.7}$$

in the sense that

$$\mathcal{E}_0(u,u) = \langle Q_E[u],[u]\rangle. \tag{2.8}$$

Note that

$$\mathcal{E}_1(u,u) = \frac{5}{3}\sum_{j=0}^{2}\langle Q_E A_j[u], A_j[u]\rangle \tag{2.9}$$

by (2.2), so that the statement $\mathcal{E}_0(u,u) = \mathcal{E}_1(u,u)$ for harmonic functions is the same as

$$TQ_E = Q_E, \tag{2.10}$$

where

$$TQ = \frac{5}{3}\sum_{j=0}^{2} A_j^* Q A_j. \tag{2.11}$$

Now the directional approximate energies are associated with other quadratic forms

$$\mathcal{E}_0^{(i)}(u,u) = \langle Q_i[u],[u]\rangle \tag{2.12}$$

for

$$Q_0 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{2.13}$$

We also have

$$\mathcal{E}_1^{(i)}(u,u) = \frac{5}{3}\sum_{j=0}^{2}\langle Q_i A_j[u], A_j[u]\rangle, \tag{2.14}$$

so that (2.5) is the same as

$$T\left(Q_i - \frac{1}{3}Q_E\right) = \frac{4}{5}\left(Q_i - \frac{1}{3}Q_E\right). \tag{2.15}$$

But the combination of (2.10) and (2.15) means that the linear transformation T has eigenvalues 1, $\frac{4}{5}$, $\frac{4}{5}$. Conversely, if we knew these were the eigenvalues of T, since we know (2.10) we could deduce (2.15) by symmetry considerations.

So Lemma 2.1 is equivalent to a statement about the eigenvalues of T. We can easily represent T by a 3×3 matrix by choosing a basis for the space of forms Q, say (Q_0, Q_1, Q_2). A routine computation shows that T is represented by

$$\frac{1}{15}\begin{pmatrix} 13 & 1 & 1 \\ 1 & 13 & 1 \\ 1 & 1 & 13 \end{pmatrix} \tag{2.16}$$

which clearly has the desired eigenvalues. This argument does not eliminate the need for a computation, but it puts the computation in an enlightening context.

In [S2] it is shown that the energy is also expressible in terms of average values

$$A_w(u) = \int u \circ F_w d\mu = 3^{-m} \int_{F_w SG} u d\mu. \tag{2.17}$$

Let

$$\tilde{\mathcal{E}}_m(u,u) = \lambda_m \sum_{w \sim_m w'} (A_w(u) - A_{w'}(u))^2 \tag{2.18}$$

for

$$\lambda_m = \frac{3}{2}\left(\left(\frac{3}{5}\right)^m - \left(\frac{3}{5}\right)^{2m}\right)^{-1}, \tag{2.19}$$

where $w \sim_m w'$ means w and w' are words of length m such that $F_w SG$ and $F_{w'} SG$ intersect at a point. Then

$$\lim_{m \to \infty} \tilde{\mathcal{E}}_m(u,u) = \mathcal{E}(u,u) \tag{2.20}$$

for all u of finite energy, and for harmonic functions $\tilde{\mathcal{E}}_m(h,h) = \mathcal{E}(h,h)$. (Note that for the limit statement we could replace λ_m by $\frac{3}{2}(\frac{5}{3})^m$.) We can easily split the approximate energies $\tilde{\mathcal{E}}_m$ into 3 directional terms according to the 3 possible orientations of the intersecting cells $F_w SG$ and $F_{w'} SG$, so

$$\tilde{\mathcal{E}}_m(u,u) = \sum \tilde{\mathcal{E}}_m^{(i)}(u,u). \tag{2.21}$$

Theorem 2.3. *For any u of finite energy,*

$$\lim_{m \to \infty} \tilde{\mathcal{E}}_m^{(i)}(u,u) = \frac{1}{3}\mathcal{E}(u,u). \tag{2.22}$$

For harmonic functions, we have more precisely

$$\widetilde{\mathcal{E}}_m^{(i)}(h,h) - \frac{1}{3}\mathcal{E}(h,h) = \frac{4^m - 3^m}{5^m - 3^m} \cdot 2(\widetilde{\mathcal{E}}_1^{(i)}(h,h) - \frac{1}{3}\mathcal{E}(h,h)). \tag{2.23}$$

Proof. As before it suffices to prove (2.23). Write

$$\widetilde{E}_m^{(i)}(h,h) = \sum (A_w(h) - A_{w'}(h))^2 \tag{2.24}$$

where the sum extends over all pairs of adjacent cells in the i direction, so that $\widetilde{\mathcal{E}}_m^{(i)}(h,h) = \lambda_m \widetilde{E}_m^{(i)}(h,h)$. The basic observation is that

$$\widetilde{E}_m^{(i)}(h,h) = \sum_{j=0}^{2} \widetilde{E}_{m-1}^{(i)}(h \circ F_j, h \circ F_j) + (A_{(i+1)i^{m-1}}(h) - A_{i(i+1)^{m-1}}(h))^2 \tag{2.25}$$

because all adjacent pairs belong to one F_jSG except for the single pair contributing the last term on the right in (2.25). Using the harmonic extension algorithm for average values (2.12) in [S2] we find that the last term in (2.25) is just

$$\left(\frac{3}{5}\right)^{2(m-1)} \widetilde{E}_1^{(i)}(h,h). \tag{2.26}$$

This leads to the expectation that

$$\widetilde{E}_m^{(i)}(h,h) = b_m \widetilde{E}_1^{(i)}(h,h) + c_m \frac{4}{25}\mathcal{E}(h,h) \tag{2.27}$$

for certain coefficients b_m, c_m to be determined, with $b_1 = 1$, $c_1 = 0$. Substituting (2.27) into (2.25) and again using the harmonic extension algorithm for average values we find after some algebraic computations the recursion relations

$$b_m = \frac{12}{25} b_{m-1} + \left(\frac{3}{5}\right)^{2(m-1)}, \tag{2.28}$$

$$c_m = \frac{1}{25} b_{m-1} + \frac{3}{5} c_{m-1}. \tag{2.29}$$

The solution to these equations is

$$b_m = \frac{25}{3}\left(\left(\frac{12}{25}\right)^m - \left(\frac{3}{5}\right)^{2m}\right) \tag{2.30}$$

$$c_m = \frac{25}{18}\left(\left(\frac{3}{5}\right)^m + \left(\frac{3}{5}\right)^{2m} - 2\left(\frac{12}{25}\right)^m\right). \tag{2.31}$$

Since $\widetilde{\mathcal{E}}_m^{(i)}(h,h) = \lambda_m \widetilde{E}_m^{(i)}(h,h)$, we find

$$(2.32) \qquad \widetilde{\mathcal{E}}_m^{(i)}(h,h) - \frac{1}{3}\mathcal{E}(h,h) = \frac{4^m - 3^m}{5^m - 3^m}\left(\frac{25}{2}\widetilde{E}_1^{(i)}(h,h) - \frac{2}{3}\mathcal{E}(h,h)\right)$$

by substituting (2.30) and (2.31) in (2.27) and simplifying. In particular

$$\widetilde{\mathcal{E}}_1^{(i)}(h,h) - \frac{1}{3}\mathcal{E}(h,h) = \frac{1}{2}\left(\frac{25}{2}\widetilde{E}_1^{(i)}(h,h) - \frac{2}{3}\mathcal{E}(h,h)\right)$$

when $m = 1$, so we obtain (2.23). □

Note that the rate of convergence in (2.23) is asymptotically the same as in (1.6), as might be expected.

3. Energy forms on SG with bilateral symmetry

In this section we examine a family of examples depending on a parameter b in $(0,\infty)$. The underlying fractal is still SG, but the energy form varies. We want a self–similar identity

$$(3.1) \qquad \mathcal{E}(u,u) = \sum_{j=0}^{2} \frac{1}{r_j}\mathcal{E}(u \circ F_j, u \circ F_j)$$

but the weights r_j are not all equal to $3/5$. We will impose the bilateral symmetry condition $r_1 = r_2$ to simplify the computations. The initial conductances defining $\mathcal{E}_0$ are also required to have this symmetry, so we may choose them as follows:

$$(3.2) \quad \mathcal{E}_0(u,u) = (u(v_0) - u(v_1))^2 + (u(v_0) - u(v_2))^2 + b(u(v_1) - u(v_2))^2.$$

It is known ([Sa] or Exercise 3.1 of [Ki2]) that $b > 0$ determines r_0 and r_1 as follows, where we introduce an additional parameter λ to simplify the expressions:

$$(3.3) \qquad r_0 = \lambda r_1\,,$$

$$(3.4) \qquad r_1 = r_2 = \frac{1+\lambda+b}{1+2\lambda+2b}\,,$$

$$(3.5) \qquad 3b^2 + 2b = \lambda^2 + 2\lambda^2 b + 2\lambda b^2\,.$$

These equations force the restriction

$$(3.6) \qquad 0 < \lambda < 3/2.$$

Notice that (3.5) is a quadratic equation in both b and λ, so we can solve in either direction

$$\lambda = \frac{-b^2 + \sqrt{b(b+1)(b^2+5b+2)}}{1+2b} \tag{3.7}$$

$$b = \frac{\lambda^2 - 1 + \sqrt{(\lambda^2-1)^2 + \lambda^2(3-2\lambda)}}{3-2\lambda} \tag{3.8}$$

to eliminate one parameter.

In view of (3.1) we define

$$\begin{aligned}\mathcal{E}_m(u,u) = \sum_{|w|=m} r_w^{-1}&((u(F_w v_0) - u(F_w v_1))^2 \\ &+ (u(F_w v_0) - u(F_w v_2))^2 + b(u(F_w v_1) - u(F_w v_2))^2),\end{aligned} \tag{3.9}$$

where $r_w = r_{w_1} \cdot r_{w_2} \cdots r_{w_m}$. A function that minimizes (3.9) for fixed boundary values is called harmonic. The equations (3.3–5) imply that $\mathcal{E}_m(u,u)$ is independent of m for harmonic functions, so $\mathcal{E}_m(u,u)$ is monotone increasing for any u, hence

$$\mathcal{E}(u,u) = \lim_{m\to\infty} \mathcal{E}_m(u,u) \tag{3.10}$$

is well-defined, and (3.1) holds. Moreover, we have an extension algorithm of the form (2.2) for harmonic functions, where the matrices have the form

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ B_3 & B_5 & B_4 \\ B_3 & B_4 & B_5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} B_3 & B_5 & B_4 \\ 0 & 1 & 0 \\ B_1 & B_2 & B_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} B_3 & B_4 & B_5 \\ B_1 & B_2 & B_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

where

$$\begin{cases} B_1 = \dfrac{1}{1+\lambda+b+2b\lambda}, \\ B_2 = \frac{1}{2}B_1(\lambda + b + 2b\lambda), \\ B_3 = B_1(1+b), \ B_4 = \frac{1}{2}B_1(\lambda+2b\lambda) - \dfrac{\lambda}{2(2\lambda+1+2b)}, \\ B_5 = \frac{1}{2}B_1(\lambda+2b\lambda) + \dfrac{\lambda}{2(2\lambda+1+2b)}. \end{cases} \tag{3.12}$$

It is natural to split the approximate energy directionally as follows:

$$\begin{cases} \mathcal{E}_m^{(0)} = \displaystyle\sum_{|w|=m} r_w^{-1}(u(F_w v_0) - u(F_w v_1))^2 \\ \mathcal{E}_m^{(1)} = b \displaystyle\sum_{|w|=m} r_w^{-1}(u(F_w v_1) - u(F_w v_2))^2 \\ \mathcal{E}_m^{(2)} = \displaystyle\sum_{|w|=m} r_w^{-1}(u(F_w v_0) - u(F_w v_2))^2 \end{cases} \tag{3.13}$$

so that (1.3) still holds, and (1.5) will define directional energies if the limits exist. We will show this is the case, and (1.7) holds with $a_0 = a_2$ by symmetry. To do this we follow the method outlined at the end of Section 2. Let T denote the operator on quadratic forms

$$TQ = \sum_{j=0}^{2} r_j^{-1} A_j^* Q A_j. \tag{3.14}$$

We use the same basis (2.13) for the space of quadratic forms. Now

$$Q_E = Q_0 + bQ_1 + Q_2 \tag{3.15}$$

and (2.10) continues to hold.

Lemma 3.1. *The eigenvalues of T are 1, μ_2, μ_3 with*

$$\mu_2 = \frac{1 + 3b + 2b^2 + 2\lambda + 4\lambda b}{1 + 4b + 4b^2 + 2\lambda + 4\lambda b} \tag{3.16}$$

$$\mu_3 = 1 - \frac{b + \frac{1}{2}}{(1 + 2\lambda + 2b)(1 + \lambda + b)} - \frac{(1 + 2\lambda + 2b)(b + \frac{1}{2})}{(1 + \lambda + b + 2\lambda b)^2 (1 + \lambda + b)} \tag{3.17}$$

with corresponding eigenvectors $(1, 1, b)$, $(-1, 1, 0)$ and $(1, 1, -\eta)$ for

$$\eta = \frac{1}{2} + \left(b + \frac{1}{2}\right)\left(\frac{1 + \lambda + b + 2\lambda b}{1 + 2\lambda + 2b}\right)^2; \tag{3.18}$$

the notation here is that (x, y, z) represents the quadratic form

$$x(u(v_0) - u(v_1))^2 + y(u(v_0) - u(v_2))^2 + z(u(v_1) - u(v_2))^2.$$

We have $|\mu_2| < 1$ and $|\mu_3| < 1$ (in fact $\frac{1}{2} < \mu_2 < 1$ and $0 < \mu_3 < 1$).

Proof. We already know $T(1, 1, b) = (1, 1, b)$. A direct calculation from (3.14) yields $T(-1, 1, 0) = \mu_2(-1, 1, 0)$ with $\mu_2 = r_0^{-1}(1 - B_3)(B_5 - B_4) + r_1^{-1}(B_3 - B_1(B_5 - B_4))$, and this simplifies to (3.16) using (3.3), (3.4), (3.5) and (3.12). Another direct calculation yields $T(0, 0, 1) = r_1^{-1}(B_1^2, B_1^2, 2B_2(1 - B_2) + \lambda^{-1}(B_5 - B_4)^2)$. Thus the eigenvalue equation $T(1, 1, -\eta) = \mu_3(1, 1, -\eta)$ can be expressed as $\mu_3(1, 1, -\eta) = (1, 1, b) - (b + \eta)T(0, 0, 1)$ and hence is equivalent to the system of equations

$$\mu_3 = 1 - \frac{b + \eta}{r_1} B_1^2 \tag{3.19}$$

$$-\mu_3 \eta = b - \frac{b + \eta}{r_1}(2B_2(1 - B_2) + \lambda^{-1}(B_5 - B_4)^2) \tag{3.20}$$

for the unknowns η and μ_3. We substitute (3.19) into (3.20) to eliminate μ_3 and obtain a quadratic equation in η. But we know that $\eta = -b$ is a solution, so we cancel the factor $\eta + b$ and obtain

$$\eta = B_1^{-2}(r_1 - 2B_2(1 - B_2) - \lambda^{-1}(B_5 - B_4)^2)$$

for the other solution, and this simplifies to (3.18). Substituting (3.18) in (3.19) yields (3.17).

It is obvious from (3.16) that $0 < \mu_2 < 1$, and from (3.17) that $\mu_3 < 1$. We obtain $\mu_2 > \frac{1}{2}$ from (3.16) by writing

$$\mu_2 = 1 - \frac{b + 2b^2}{1 + 4b + 4b^2 + 2\lambda + 4\lambda b}$$

and observing $b + 2b^2 < \frac{1}{2}(1 + 4b + 4b^2 + 2\lambda + 4\lambda b)$. To show $\mu_3 > 0$ we rearrange terms in (3.17) to reduce it to the inequality

$$\left(b + \frac{1}{2}\right)[(1 + \lambda + b + 2\lambda b)^2 + (1 + 2\lambda + 2b)^2] < (1 + \lambda + b)(1 + 2\lambda + 2b)(1 + \lambda + b + 2\lambda b)^2. \tag{3.21}$$

Now (3.21) is implied by

$$\left(b + \frac{1}{2}\right)[(1 + \lambda + b + 2\lambda b)^2 + (1 + 2\lambda + 2b)^2] < (1 + \lambda + b)(2b + 1)(1 + \lambda + b + 2\lambda b)^2$$

which is equivalent, after canceling the $b + \frac{1}{2}$ factor, to

$$(1 + \lambda + b + 2\lambda b)^2 + (1 + 2\lambda + 2b)^2 < (2 + 2\lambda + 2b)(1 + \lambda + b + 2\lambda b)^2$$

which is the same as

$$(1 + 2\lambda + 2b)^2 < (1 + 2\lambda + 2b)(1 + \lambda + b + 2\lambda b)^2$$

or

$$(1 + 2\lambda + 2b) < (1 + \lambda + b + 2\lambda b)^2,$$

which is obvious.

There is another proof that $\mu_3 > 0$. Suppose for a moment $\mu_3 \leq 0$. Let Q be the eigenvector of T of the form $(1, 1, -\eta)$ and h be a symmetric harmonic function with boundary values $h(v_1) = h(v_2) = 0$, $h(v_0) = 1$. We will write $Q(h)$

for $\langle Q[h],[h]\rangle$). Then $Q(h)=2$. By our assumption $TQ(h)=\mu_3 Q(h)\le 0$. Clearly $Q(h\circ F_0)>0$ and so $Q(h\circ F_1)=Q(h\circ F_2)<0$. Then $Q(h\circ F_1)<0$ easily implies $\eta>1$ since $0=h\circ F_1(v_1)<h\circ F_1(v_2)<h\circ F_1(v_0)$.

Let now h be a skew symmetric harmonic function with boundary values $h(v_1)=1$, $h(v_2)=-1$, $h(v_0)=0$. Then $Q(h)=2-4\eta<0$ by the previous paragraph. By our assumption $TQ(h)=\mu_3 Q(h)\ge 0$. Clearly $Q(h\circ F_0)<0$ and so $Q(h\circ F_1)=Q(h\circ F_2)>0$. Let $x=h\circ F_1(v_0)$. We have $0<x<1$ and so $Q(h\circ F_1)=1-2x+2x^2-\eta<0$. This is a contradiction and so $\mu_3>0$. □

Note that $\mu_2\to 1$ as $b\to 0$ (hence $\lambda\to 0$) and $\mu_3\to 1$ as $b\to\infty$ (hence $\lambda\to 3/2$). This means that the rate of convergence of the directional energies to their limits becomes extremely slow as we approach either extreme.

Theorem 3.2. *For any function u of finite energy (3.10),*

$$\lim_{m\to\infty}\mathcal{E}_m^{(j)}(u,u)=a_j\mathcal{E}(u,u) \tag{3.22}$$

for

$$\begin{cases} a_0=\dfrac{b}{\eta+b}=\dfrac{2b}{2b+1}\left(1+\left(\dfrac{1+\lambda+b+2\lambda b}{1+2\lambda+2b}\right)^2\right)^{-1}, \\ a_1=a_2=\dfrac{\eta}{2(\eta+b)}. \end{cases} \tag{3.23}$$

Proof. We write $(0,0,b)=a_0(1,1,b)-a_0(1,1,-\eta)$ for $a_0=b/(\eta+b)$ hence $T^m(0,0,b)\to a_0(1,1,b)$ as $m\to\infty$. Similarly $(1,0,0)=a_1(1,1,b)+(a_1 b/\eta)(1,1,-\eta)-\frac{1}{2}(-1,1,0)$ for $a_1=\eta/2(\eta+b)$, hence $T^m(1,0,0)\to a_1(1,1,b)$. □

Figure 3 shows the graphs of a_0 and a_1 as functions of b. Note that there is one other value of b, approximately 1.65249025 where $a_0=a_1=1/3$. Also, as $b\to 0$, $a_0\to 1/2$ and $a_1\to 0$, while when $b\to\infty$, $a_0\to 1/5$ and $a_2\to 2/5$. These facts are easily seen from (3.22).

4. The General Case

Here we generalize for other fractals the “enlightening” approach to the partition of energy presented in the middle of Section 2. Suppose we have a p.c.f. self–similar fractal K as defined in [Ki1,Ki2]. This means, in particular, that K is an (abstract) compact set such that

$$K=\bigcup_{i=1}^{N}F_iK$$

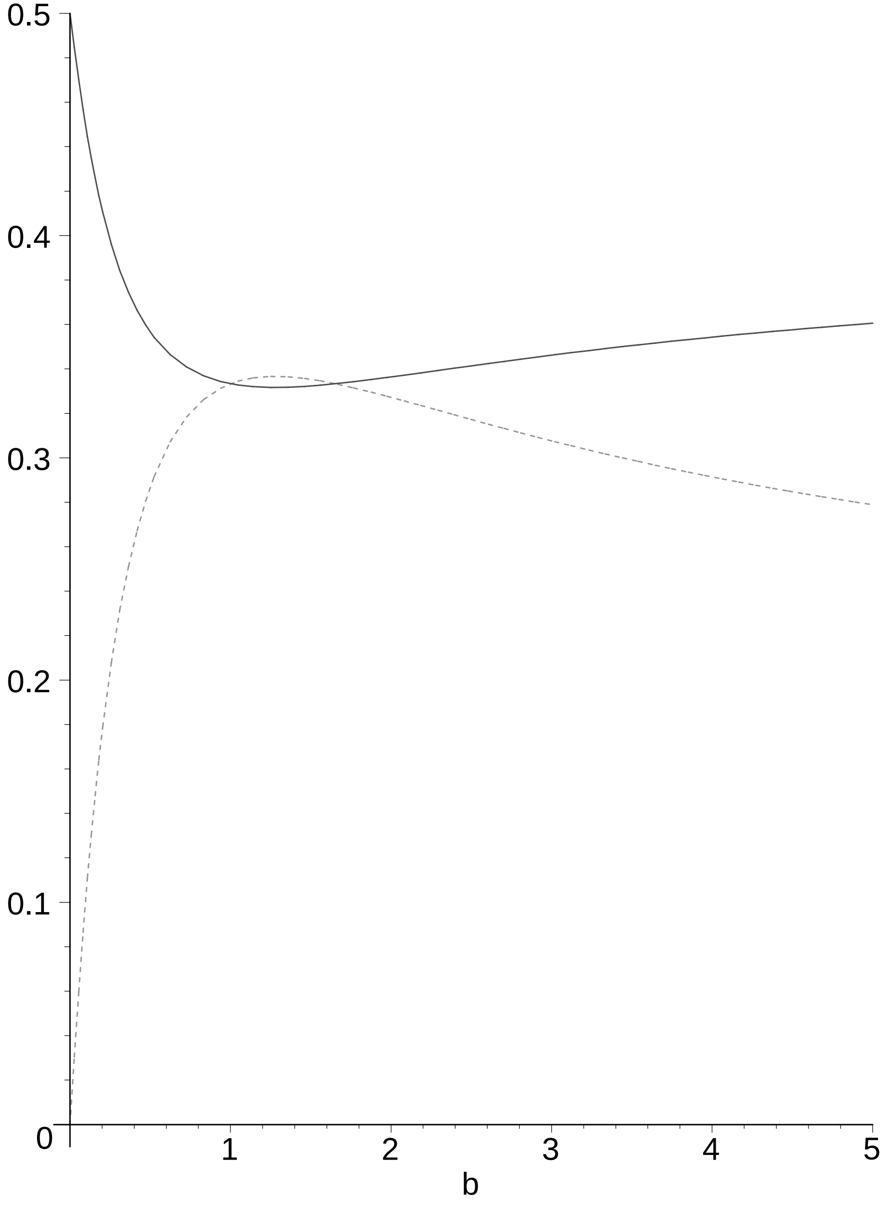

FIGURE 2. The graphs of a_0 (solid line) and a_1 (dotted line) as a function of the parameter b in the range $0 < b < 5$.

where F_i are continuous injections, which are contractive in most examples, and there is a finite subset $\partial K = V_0 = \{v_1, \dots, v_{N_0}\}$, called the *boundary* of K, such that

$$F_w K \cap F_{w'} K \subseteq F_w V_0 \cap F_{w'} V_0$$

for F_w and $F_{w'}$ distinct.

Suppose also that we have a local regular self–similar Dirichlet form $\mathcal{E}$ on K. This means, similarly to (1.1) and (1.2), that

$$\mathcal{E}(u,u) = \lim_{m\to\infty} \mathcal{E}_m(u,u) \tag{4.1}$$

where

$$\mathcal{E}_0(u,u) = \sum_{j\neq k} c_{jk}(u(v_j) - u(v_k))^2 \tag{4.2}$$

$$\mathcal{E}_m(u,u) = \sum_{|w|=m} r_w^{-1}\mathcal{E}_0(u\circ F_w, u\circ F_w) \tag{4.3}$$

for a choice of $r_j \in (0,1)$ with the "decimation property" that $\mathcal{E}_m(u,u)$ is independent of m for harmonic functions (energy minimizers given the N_0 boundary values on V_0). The self–similar identity

$$\mathcal{E}(u,u) = \sum_{j=1}^{N} r_j^{-1}\mathcal{E}(u\circ F_j, u\circ F_j)$$

is then an immediate consequence of (4.1) and (4.3). We assume $c_{jk} \geq 0$ and $\mathcal{E}_0(u,u) > 0$ unless u is constant on V_0.

It is still not known in general under what conditions such a Dirichlet form exists, but there are sufficient conditions and numerous examples (see [Ki1, Ki2, M1, M2, Sa] and references therein). It should be noted that the choice of the conductance coefficients c_{jk} and the energy ratios r_j is a very delicate matter.

Let $A_1, \dots, A_N$ denote the harmonic extension matrices, so that

$$u|_{F_i V_0} = A_i u|_{V_0}$$

for any harmonic function u. These matrices are obtained by solving the system of linear equations

$$\sum_{i=1}^{N}\sum_{j\neq k} r_i^{-1} c_{jk}(u(F_i v_j) - u(F_i v_k)) = 0.$$

The constant vector is invariant under all the matrices A_i, and we need to factor out this trivial common eigenvector. Let W_0 denote the space of functions on V_0, let W denote the quotient space of W_0 modulo constants, and let M_i denote the induced action of A_i on W. Note that W is a vector space of dimension $N_0 - 1$, and we could represent M_i by a matrix of size $(N_0 - 1) \times (N_0 - 1)$ by choosing a basis for W. Let $M_w = M_{w_1} \circ M_{w_2} \circ \cdots \circ M_{w_m}$.

Hypothesis 4.1. *Assume the following span–irreducibility condition:*

(4.4) *For every nonzero $x \in W$ there exists $m \geq 1$ such that the span of $\{M_w x\}_{|w|=m}$ is all of W.*

Note that routine compactness arguments show that we can find a single m for all x. Also note that the span–irreducibility condition implies that $M_1, \ldots, M_N$ are irreducible (they do not have a proper common invariant subspace), but the converse is not true in general.

Lemma 4.2. *Condition* (4.4) *holds if there is no proper subspace of W invariant under all M_i, and there is at least one point $v \in V_0$ which is fixed by one of the maps F_i.*

Proof. Suppose $F_i(v) = v$. Consider the subspace of harmonic functions that vanish at v. This subspace is invariant under A_i, and all the entries of A_i restricted to this subspace are nonnegative, with at least one positive. Note that this subspace can be naturally identified with W. Then, by the Perron–Frobenius Theorem, there is a positive eigenvalue λ of M_i such that $|\lambda'| < \lambda < 1$ for any other eigenvalue λ' of M_i. Let W_λ be the eigenspace of M_i corresponding to λ (there is no nondiagonal Jordan block corresponding to λ).

If $x \in W_\lambda$, then the spaces $W_m = \text{Span}\,\{M_w x\}_{|w|=m}$ are increasing and so we have $W_m = W$ for some $m \leq N_0 - 1$, since there are no proper common invariant subspaces. So (4.4) holds with $m = N_0 - 1$ for $x \in W_\lambda$.

For a general nonzero vector $x \in W$, let $W' = \text{Closure} \bigcup_{m \geq 0} W_m$, where W_m is as in the previous paragraph. Then Span $W' = W$ since there are no proper invariant subspaces. Also if $y \in W'$ then for all j we have

$$\lim_{m \to \infty} \frac{M_j^m y}{\|M_j^m y\|} \in W'.$$

Then it is easy to see that $W' \cap W_\lambda \neq 0$. A simple continuity argument implies (4.4). □

The energy form (4.2) can be regarded as a symmetric bilinear form on W_0 annihilating constants, as can its directional components

$$\mathcal{E}_0^{(jk)}(u, u) = c_{jk}(u(v_j) - u(v_k))^2.$$

We can reduce from W_0 to W and express these as nonnegative quadratic forms Q_E and Q_{jk} satisfying

$$Q_E = \sum_{j \neq k} Q_{jk}.$$

We define approximate directional energies by

$$\mathcal{E}_m^{(jk)}(u,u) = \sum_{|w|=m} r_w^{-1}\mathcal{E}_0^{(jk)}(u \circ F_w, u \circ F_w)$$

and directional energies

$$\mathcal{E}^{(jk)}(u,u) = \lim_{m\to\infty} \mathcal{E}_m^{(jk)}(u,u) \tag{4.5}$$

if the limit exists (similarly to (1.5)). Then the analog of Theorem 2.2 is as follows.

Theorem 4.3. *Under Hypothesis 4.1, the limits* (4.5) *exist and there exist positive constants* α_{jk} *such that*

$$\mathcal{E}^{(jk)}(u,u) = \alpha_{jk}\mathcal{E}(u,u)$$

for all functions u *of finite energy.*

We define an operator T on quadratic forms on W by

$$TQ = \sum_{j=1}^{N} r_j^{-1} M_j^* Q M_j. \tag{4.6}$$

Then the decimation property that $\mathcal{E}_m(u,u)$ is independent for m when u is harmonic is equivalent to

$$TQ_E = Q_E. \tag{4.7}$$

Theorem 4.3 follows from the next lemma by the same proof as Theorem 2.2.

Lemma 4.4. *Under Hypothesis 4.1, there exist positive constants* α_{jk} *such that*

$$\lim_{m\to\infty} T^m Q_{jk} = \alpha_{jk} Q_E.$$

Lemma 4.4 follows from Theorem 4.6, which is a special case of a kind of Perron–Frobenius Theorem for operators of the form (4.6). We begin with a simple result of Kusuoka [Ku].

Proposition 4.5. *If* T *is an operator of the form* (4.6) *for* $M_1, \ldots, M_N$ *irreducible, then there is a unique non–negative definite eigenvector. Moreover the eigenvector is positive definite and the eigenvalue is positive.*

Without loss of generality we may assume the eigenvalue is 1, and we denote the eigenvector by Q_E to conform to (4.7). The next theorem is not new and can be deduced from different generalizations of the Perron–Frobenius Theorem. We give a concise proof here for the sake of completeness.

Theorem 4.6. *If $M_1, \dots, M_N$ satisfy the span–irreducibility condition* (4.4)*, then* 1 *is a simple eigenvalue of T and every other eigenvalue λ satisfies $|\lambda| < 1$. Moreover, for any nonzero Q that is nonnegative definite there exists a positive constant $\alpha(Q)$ such that*

$$\lim_{m\to\infty} T^m(Q) = \alpha(Q)Q_E. \tag{4.8}$$

Proof. We will write $A > B$ ($A \geq B$) if $A-B$ is positive–definite (nonnegative–definite) matrix. Because $Q_E > 0$ by Proposition 4.5, for any real Q there exists a constant c such that $Q \leq cQ_E$, hence $T^mQ \leq cQ_E$ for any m. Thus T cannot have an eigenvalue λ with $|\lambda| > 1$, nor a nondiagonal Jordan block corresponding to an eigenvalue with $|\lambda| = 1$. This does not use (4.4).

Without loss of generality (pass to a power of T) we may assume that $m = 1$ in (4.4). We claim that for any nonzero symmetric Q satisfying $Q \geq 0$ we have $TQ > 0$. Indeed, for any $x \neq 0$,

$$\langle TQx, x\rangle = \sum_{j=1}^{N} r_j^{-1}\langle QM_jx, M_jx\rangle > 0$$

because $\{M_jx\}$ spans the whole space.

Suppose $TQ_\lambda = \lambda Q_\lambda$ with $|\lambda| = 1$ and Q_λ is not a multiple of Q_E. Denote by $\overline{Q}_\lambda$ the matrix whose entries are complex adjoint of the entries of Q_λ. Then $Q_\lambda + \overline{Q}_\lambda$ is real and we can assume it is not nonpositive. Since $Q_E - \alpha(Q_\lambda + \overline{Q}_\lambda) \geq 0$ for α small enough, we may define ε so that $Q_E - \varepsilon(Q_\lambda + \overline{Q}_\lambda) \geq 0$ and ε achieves the maximum value subject to this condition.

Then $T(Q_E - \varepsilon(Q_\lambda + \overline{Q}_\lambda)) > 0$ by the claim, hence we have

$$Q_E - \varepsilon(\lambda Q_\lambda + \overline{\lambda Q_\lambda}) \geq \delta Q_E$$

for some $\delta > 0$, and more generally

$$Q_E - \varepsilon(\lambda^n Q_\lambda + \overline{\lambda^n Q_\lambda}) \geq \delta Q_E$$

for any $n \geq 1$. By choosing n so that λ^n is sufficiently close to 1 we contradict the maximality of $|\varepsilon|$.

Finally, if $Q \geq 0$ is nonzero then there is $\delta > 0$ such $T^nQ > \delta Q_E$ for all $n \geq 1$. Thus we can write Q as a linear combination of eigenvectors and associated eigenvectors of T where the coefficient $\alpha(Q)$ of Q_E is positive. This yields (4.8). ◻

Theorem 4.7. *For a given set of positive energy weights r_j there exists at most one, up to a constant multiple, local regular self–similar Dirichlet form provided the span–irreducibility condition* (4.4) *is satisfied for at least one such form.*

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be two such Dirichlet forms, and Q_E and Q'_E be the corresponding quadratic forms. Suppose that (4.4) is satisfied for $\mathcal{E}$, and define T in terms of $\mathcal{E}$. Then Theorem 4.6 implies that $\lim_{m\to\infty} T^m(Q'_E) = \alpha Q_E$. Hence for any piecewise $\mathcal{E}$–harmonic function f we have $\mathcal{E}'(f,f) = \alpha\mathcal{E}(f,f)$. Since piecewise harmonic functions are dense in $C(K)$ (see [Ki1,Ki2]), we have $\mathcal{E}' = \alpha\mathcal{E}$. □

4.8. ***Remark.*** Results similar to Theorems 4.6 and 4.7 have appeared in the literature several times, for example in [P]. These theorems give so called "existence implies stability implies uniqueness" arguments provided some irreducibility conditions hold. Our approach is somewhat simpler and more straightforward, although the absence of invariant subspaces may be difficult to verify in some situations. We do not require that $K \subset \mathbb{R}^n$ with F_i linear contractions, but this assumption may not be crucial anyway. Perhaps it is more important that we do not require that every point of V_0 is a fixed point of some F_i (so called "essential fixed points"). We will abbreviate this condition as the EFP condition. Below we give a simple example where this condition is not satisfied, and there are many more examples.

There is an important relation of our work to the existence and uniqueness results by V. Metz [M1,M2] and C. Sabot [Sa] (see also references in these papers). Namely, the map T is the derivative at Q_E of a nonlinear renormalization operator, usually called $\mathcal{F}_r$ or Λ, for which Q_E is a fixed point (see [Ki1,Ki2,M1,M2,Sa]). It is shown in [M2] that the uniqueness holds if for any nonzero $Q \geq 0$ we have $T^n Q > 0$ for large enough n. It is easy to see that this is equivalent to our span–irreducibility condition (4.4). However the condition $T^n Q > 0$ is difficult to verify mainly because the cone of nonnegative operators is more difficult to characterize in the context of fractals than the cone of Dirichlet forms. So [M2] also contains a very useful uniqueness condition in terms of Dirichlet forms. However [M2] assumes the EFP condition, and it is not clear how crucial this condition is to the results there.

The span–irreducibility condition (4.4) is similar to the conditions studied in [KK] under the EFP condition. If F_i has a fixed point v in V_0, then A_i has at least one Perron–Frobenius eigenvector that vanishes at v, as described in the proof of Lemma 4.2. Denote the collection of these eigenvectors for all i by I_{PF}. By factoring out constants, we may assume that $I_{PF} \subset W$. The assumptions in [KK] included some symmetry assumptions and that Span $I_{PF} = W$. The following lemma is easy to prove similarly to Lemma 4.2.

Lemma 4.8. *Suppose* Span $I_{PF} = W$. *Condition* (4.4) *holds if, for any subset* I'_{PF} *of* I_{PF}, Span I'_{PF} *is not invariant under all* M_i *unless* Span $I'_{PF} = W$.

It is easy to see that in the well known example of the Vicsek set we have Span $I_{PF} = W$ but the span–irreducibility fails, as does the energy partition result. It is known from the works of V. Metz that the self–similar Dirichlet form is not unique in this case. Below we gave an example (without the EPF condition) where the span–irreducibility, and hence the uniqueness and energy partition, hold.

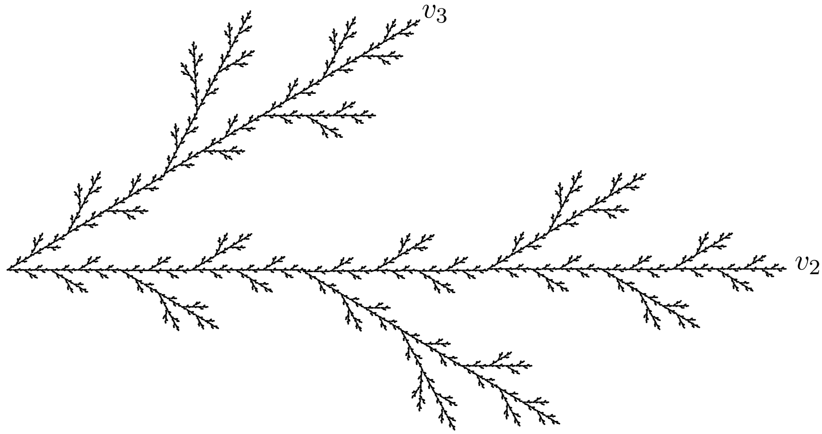

FIGURE 3. Hata's tree.

Example 4.9. Suppose β is a complex number such that $|\beta| < 1$ and $|1-\beta| < 1$. We define two contractions in $\mathbb{C}$ by $F_1(z) = \beta\overline{z}$ and $F_2(z) = (1-|\beta|^2)\overline{z}+|\beta|^2$. Then the iterated function system $\{F_1, F_2\}$ defines a p.c.f fractal called Hata's tree (see Example 8.4 in [Ki1]) which has three boundary points v_1, v_2, v_3, but only two of them, $v_1 = 0$ and $v_2 = 1$, are fixed points. According to [Ki1], there is a self–similar Dirichlet form with $r_1 = \alpha$ and $r_2 = 1 - \alpha^2$ for any $0 < \alpha < 1$. The matrix of the Dirichlet form $\mathcal{E}_0$ is

$$Q_E = \begin{pmatrix} 1+\alpha^{-1} & -1 & -\alpha^{-1} \\ -1 & 1 & 0 \\ -\alpha^{-1} & 0 & \alpha^{-1} \end{pmatrix}$$

and the harmonic matrices are

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ a & 1-a & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} a & 1-a & 0 \\ 0 & 1 & 0 \\ a & 1-a & 0 \end{pmatrix}$$

where $0 < a < 1$ depends on α. The span–irreducibility condition is easy to verify. Hence for each α the Dirichlet form is unique and Theorems 4.6 and 4.7 hold.

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