ELECTRICAL RESISTANCE OF N-GASKET FRACTAL NETWORKS

BRIGHID BOYLE, KRISTIN CEKALA, DAVID FERRONE, NEIL RIFKIN, AND ALEXANDER TEPLYAEV

Abstract. We study self-similar local regular Dirichlet, or energy, forms on a class of fractal N-gaskets, which are generalizations of polygaskets. This is directly related to self-similar diffusions and resistor networks (electrical circuits). We prove existence and uniqueness, and also obtain explicit formulas for scaling factors and resistances (transition probabilities). We also study asymptotic behavior of these quantiles as the number of "sides" N of an N-gasket tends to infinity.

Contents

Introduction

This paper is a part of a relatively, but now well established, new field of analysis and probability on fractals, see [1, 3, 4, 16, 17, 18, 24, 25, 31, 35, 38, 39, 40, and references therein] for a sample of mathematical literature on the analysis on fractals. Furthermore, many of the questions addressed here are related to the general Dirichlet form theory; for further information on this subject the reader can refer to the now classical books [5, 10]. To be more precise, we are interested in local regular self-similar Dirichlet, or energy, forms on a subclass of finitely ramified self-similar fractals. Non local Dirichlet forms on fractals have been also extensively studied, see for instance [13, 43, and references therein].

Roughly speaking, self-similar fractals are objects that can be divided, with infinite detail, into smaller objects of similar shape. A snowflake is a classic example. The properties and physical applications of various types of these objects have been

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the subject of many studies in physics, of which we mention only a few [2, 11, 28, 29, 32]. Fractal antennas were studied and successfully tested, in particular, in some wireless devices [14, 26, 27].

One particular type of self-similar fractal objects, known as polygaskets, are formed by dividing a regular polygon into smaller regular polygons of the same shape, arranged in a circular pattern [1, 33, 35, 36, 38, 41, 42]. If a polygasket is finitely ramified (that is, point connected), then it belongs to the class of so called nested fractals [20, 22, 38]. The N-gaskets belong to the larger class of so called p.c.f. self-similar sets [17, 18], and are generalizations of finitely ramified polygaskets.

In this study we consider self-similar fractal resistor networks (electrical circuits) or, equivalently, a self-similar energy (Dirichlet) form on an N-gasket. Using a combination of various methods of mathematical analysis, some of which come from electric engineering (see Appendix), we construct networks with the property that every resistor in a network of a given order is a scalar multiple of the corresponding resistor in the next-order network, this scalar being constant across the fractal. This allows to translate the resistance of these fractal electrical networks into conventional, easily manipulated networks. We prove existence and uniqueness, up to a multiplicative constant, of a self-similar resistance (Dirichlet, energy) form on any N-gasket. In general this is a difficult and delicate question [12, 21, 22, 23, 30]. We derive explicit formulas which uniquely determine all the resistances and scaling constant for every N-gasket. Furthermore, we explore the behavior of these constants as certain parameters are allowed to tend to infinity, thus capturing the general properties of the current flow in an N-gasket with large N.

This paper is organized as follows. In Section 1 we define N-gaskets and state our main objectives. In Section 2 we transform the self-similar structure of an N-gasket in order to simplify the problem and to be able to use some ideas from electrical engineering. In Section 3 we compute the resistance scaling factor c, which is one of our main results, and also the resistances in the transformed network. In Section 4 we use the results of the previous sections to compute the resistances in the original problem we deal with. In Section 5 we study various asymptotic behaviors of the resistance scaling factor c and of resistances as parameters of an N-gasket approach infinity. In particular, we compute the asymptotic of the dimension in effective resistance metric and asymptotic spectral dimension in Corollary 5.4. In Appendix we provide a brief description of the relevant electrical circuits formalism, prove an auxiliary result which is used in the main body of the paper, describe relation of or results to the theory of random walks, and give some related references.

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1. Definitions and examples

An N-gasket is a self-similar fractal F which is a compact metric space made of N scaled copies F_1, ..., F_N of itself, that is

$$F = \bigcup_{j=1}^{N} F_j.$$

These copies F_j are called depth-1 cells, or 1-cells for short. The self-similarity means that there are contractive homeomorphisms

$$\psi_j : F \to F_j$$

for each $j = 1, ..., N$.

By definition, the so called boundary of the N-gasket consists of N points v_1, ..., v_N, one in each 1-cell, with the property $\psi_j(v_j) = v_j$. The boundary of the k-th scaled copy F_k of an N-gasket consists of points $v_{k,1}$, ..., $v_{k,N}$, numbered in such way that

$$v_{k,j} = \psi_k(v_j).$$

In particular, $v_k = v_{k,k}$.

Notation 1.1. *When numbering cells in an N-gasket we always use cyclic notation* mod (N)*, that is $F_{N+1} = F_1$, $v_{N+1} = v_1$ etc.*

Up a homeomorphism an N-gasket is defined by three numbers N_1, N_2, N_3 with $N_1 + N_2 + N_3 = N$. The most important property of an N-gasket is that

$$F_k \cap F_{k+1} = v_{k,k+N_3},$$
$$F_k \cap F_{k-1} = v_{k,k+N-N_2},$$

and the depth-1 cells are otherwise disjoint (note that we use cyclic notation as in Notation 1.1). Thus, we have identifications

$$v_{k,k+N_3} = v_{k+1,k+1+N-N_2}.$$

This means that each k-th scaled copy F_k of the N-gasket is connected to the other scaled copies via its boundary points v'_{N_3+k} and v'_{N-N_2+k} if we denote $v_{k,j} = v'_j$. Thus, an N-gasket with these connectivity properties we will sometimes call the (N_1, N_2, N_3)-gasket. In Figure 1 one can see that the Sierpiński gasket is a $(1, 1, 1)$-gasket, the pentagasket is a $(1, 2, 2)$-gasket, and the hexagasket is a $(2, 2, 2)$-gasket. The so called fractal cut square and the fractal red cross in Figure 2 are a $(2, 1, 1)$-gasket and a $(1, 1, 2)$-gasket respectively.

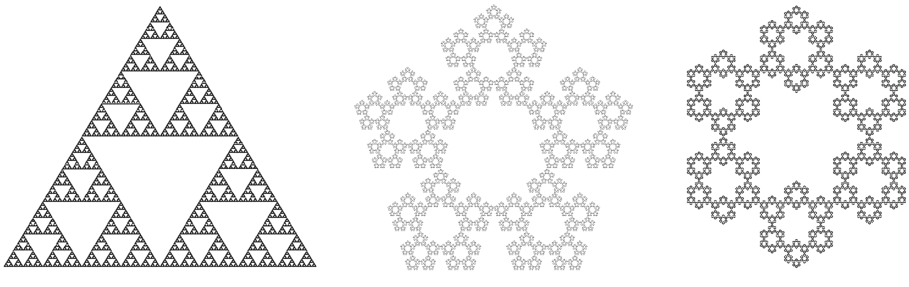

FIGURE 1. Examples of polygaskets: Sierpiński gasket, pentagasket and hexagasket, which are $(1, 1, 1)$-gasket, $(1, 2, 2)$-gasket, and $(2, 2, 2)$-gasket respectively.

One can see from general theory [15, 16] that an N-gasket exists for any triple (N_1, N_2, N_3) of positive integers with $N = N_1 + N_2 + N_3$. An N-gasket can be easily constructed as a factor space of the one-sided shift space $\{1, ..., N\}^{\mathbb{N}}$, as in

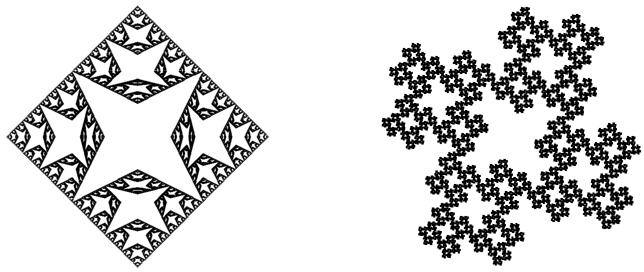

FIGURE 2. Examples of 4-gaskets: fractal cut square and fractal red cross, which are $(2, 1, 1)$-gasket and $(1, 1, 2)$-gasket respectively.

the appendix to [16]. Moreover, an N-gasket is defined uniquely up to a homeomorphism by the triple (N_1, N_2, N_3) except that an (N_1, N_2, N_3)-gasket and an (N_1, N_3, N_2)-gasket are homeomorphic via the obvious "reflection" map.

Note that any polygasket, and any nested fractal in general, by definition has a dihedral symmetry group. It is easy to see from the definition that for any N-gasket there is a homeomorphism ρ such that

$$\rho \circ \psi_j = \psi_{j+1} \circ \rho.$$

In particular, $\rho(F_j) = F_{j+1}$, $\rho(v_j) = v_{j+1}$, and $\{\rho^k\}_{k=1}^N$ is a cyclic group of N elements isomorphic to $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. Without loss of generality we can assume that ρ is an isometry. An N-gasket can be embedded in $\mathbb{R}^d$ in such a way that this isometry ρ is a rotation by $\frac{2\pi}{N}$. Thus, we have the following proposition.

Proposition 1.2. *An N-gasket has the dihedral symmetry group D_N if and only if $N_2 = N_3$. Otherwise the symmetry group is the cyclic group $\mathbb{Z}_N$ of N elements.*

We will denote the symmetry group of an N-gasket by $\mathcal{G}$.

We are interested in the existence and uniqueness, up to a multiplicative constant, of a self-similar local regular $\mathcal{G}$-invariant resistance (Dirichlet, energy) form $\mathcal{E}$ on any N-gasket. By the general results [4, 12, 16, 18, 22, 23, 30, 38] it is enough to prove the existence and uniqueness of a scaling constant c and resistances R_k, $k = 1, ..., N-1$, such that the following property holds.

First, consider a resistor network with vertices $\{v_i\}_{i=1}^N$ such that each pair v_i and v_{i+k} is connected by a resistance R_k. We will call this network the 0-depth network. Note that we use cyclic notation $\mod (N)$ for indices. Let R'_k be the effective resistance between v_i and v_{i+k}, which obviously does not depend on i, in this 0-depth network. Second, consider a resistor network with vertices $\{v_{i,j}\}_{i,j=1}^N$ such that each pair $v_{i,j}$ and $v_{i,j+k}$ is connected by a resistance R_k, and pairs of distinct vertices in different 1-cells are not directly connected. We will call this network the 1-depth network. Note that we have identification of vertices $v_{k,k+N_3} = v_{k+1,k+1+N-N_2}$. Let R''_k be the effective resistance between v_i and v_{i+k} in this second 1-depth network, which again does not depend on i. These two networks are illustrated in Figures 3 and 4. The existence and uniqueness of a self-similar local regular $\mathcal{G}$-invariant resistance (Dirichlet) form $\mathcal{E}$ on F is equivalent to the existence and uniqueness of a resistance scaling constant c and resistances R_k, $k = 1, ..., N-1$, such that

$$R''_k = cR'_k \tag{1.1}$$

for all $k = 1, ..., N$.

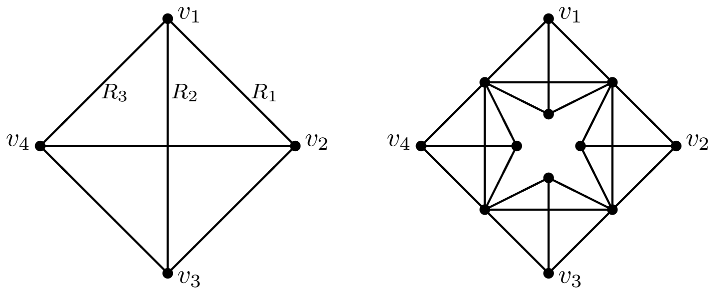

FIGURE 3. The 0-depth and 1-depth networks in the case of the fractal cut square.

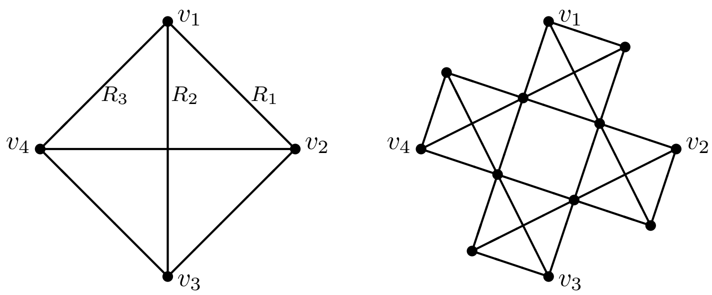

FIGURE 4. The 0-depth and 1-depth networks in the case of the fractal red cross.

In other terms, let

$$V_0 = \{v_i\}_{i=1}^N \subset V_1 = \{v_{i,j}\}_{i,j=1}^N.$$

On V_0 we define a finite dimensional $\mathcal{G}$-symmetric Dirichlet form

$$\mathcal{E}_0(f, f) = \sum_{i \neq j} \frac{1}{R_{i-j}} (f(v_i) - f(v_j))^2,$$

and then on V_1 we define a finite dimensional $\mathcal{G}$-symmetric Dirichlet form Dirichlet form

$$\mathcal{E}_1(g, g) = \sum_{k=1}^N \sum_{i \neq j} \frac{1}{R_{i-j}} (g(v_{k,i}) - g(v_{k,j}))^2.$$

Then our main equation, which is equivalent to (1.1), is

$$\mathcal{E}_0 = cTr_{V_0} \mathcal{E}_1, \tag{1.2}$$

that is $\mathcal{E}_0$ is the same as the trace of $c\mathcal{E}_1$ on V_0. The trace of Dirichlet forms on subsets is defined in the Appendix.

Equations (1.1) and (1.2) allow to define a local regular self-similar Dirichlet form on F by

$$\mathcal{E}(f, f) = \lim_{n \to \infty} c^n \sum_{w_1, \ldots, w_n = 1}^N \mathcal{E}(f \circ \psi_{w_n} \ldots \circ \psi_{w_1}, f \circ \psi_{w_n} \ldots \circ \psi_{w_1})$$

for any f in the domain of the form. The reader can find the general theory of Dirichlet forms in [5, 10], and the theory of self-similar Dirichlet forms on fractals in [4, 18, 38]. According to this theory, on F there exists a densely defined self-adjoint operator Δ, called the Dirichlet self-similar Laplacian on F, such that

$$\mathcal{E}(f,f) = \int_F f\Delta f \, d\mu$$

for any function in the domain of the Δ that vanishes on the boundary of F. Here μ is the unique self-similar balanced probability measure on F, which has the property that

$$\int_F f \, d\mu = \frac{1}{N} \sum_{i=1}^{N} \int_F f \circ \psi_i \, d\mu$$

for any bounded Borel measurable function f.

For any resistance form there correspond the so called effective resistance metric

$$R_{eff}(p,q) = \sup \left\{ \frac{\big(u(p) - u(q)\big)^2}{\mathcal{E}(u,u)} : u \in \ Dom\mathcal{E}, \mathcal{E}(u,u) > 0 \right\}.$$

This metric is studied in detail in [17, 18]. In particular, it is not self-similar, but asymptotically self-similar with scaling constant c. The effective resistance metric is not related to a particular embedding of the fractal into $\mathbb{R}^n$, but rather to its intrinsic structure.

Remark 1.3. *The resistance scaling constant c is important, in particular, because it allows to compute the Hausdorff dimension of the fractal with respect to the effective resistance metric. According to* [17, 18, 19]*, the Hausdorff dimension of the N-gaskets with respect to the effective resistance metric is*

$$d_H = \frac{\log N}{\log c}.$$

Note that this dimension has no relation with the embedding of the fractals into $\mathbb{R}^2$ *in Figures 1 and 2, and it does not correspond to the "usual" dimension of self-similar fractals in* $\mathbb{R}^n$*. It rather depends on the "inner" structure of the fractal, and on the way in which parts of the fractal are connected.*

By definition, the so called spectral dimension of the Laplacian is equal to

$$d_S = 2 \lim_{x \to \infty} \frac{\log(\rho(x))}{\log(x)},$$

if the limit exists, where $\rho(x)$ *is the eigenvalue counting function of the Laplacian. The spectral dimension is not a dimension in the usual sense of topology or geometric measure theory, but just the exponent that determines rate of growth of the eigenvalue counting function of the Laplacian. However it is called "dimension" because in the case of the usual Laplacian on a compact Riemannian d-dimensional manifold or a domain in* $\mathbb{R}^d$ *the spectral dimension is equal to the topological and metric dimension d. According to* [18, 19]*, for the standard Laplacian on a p.c.f. self-similar set we can compute the spectral dimension by the formula*

$$d_S = \frac{2d_H}{d_H + 1} = \frac{2 \log N}{\log(Nc)}.$$

According to [37, 38] *this formula shows that the Laplacian is an operator of order* $d_H + 1$.

Note that the spectral dimension d_S is not equal to the Hausdorff dimension d_H unless both are one. These dimensions for any N-gasket can be computed using Theorem 2, and their asymptotic behaviors under different assumptions are given in Corollary 5.4.

2. Transformed N-gaskets

Equivalent equations (1.1) and (1.2) can be considered a nonlinear $(N-1)$-dimensional eigenvalue problem, which is difficult to solve in general [12, 22, 23, 30]. To achieve our results we have to change self-similarity structure of F and apply various transformations to the networks involved.

We define

$$\widetilde{\psi}_j = \rho^{j-1} \circ \psi_1$$

and note that by definition $\widetilde{\psi}_j(F) = F_j$ and $\widetilde{\psi}_j(v_k) = v_{j+k-1}$. This means, in particular, that

$$\widetilde{\psi}_k(v_1) = v_k,$$
$$\widetilde{\psi}_k(v_{N_3+1}) = v_{k,k+N_3},$$
$$\widetilde{\psi}_k(v_{N-N_2+1}) = v_{k,k+N-N_2}.$$

It is important that the vertices in the left had side of these three formulas do not depend on k. Thus, we can consider the three vertices $v_1, v_{N_3+1}, v_{N-N_2+1}$ as the boundary of F if we use the self-similar structure of contractions $\widetilde{\psi}_1, ..., \widetilde{\psi}_N$. For simplicity we will denote

$$q_1 = v_1,$$
$$q_2 = v_{N_3+1},$$
$$q_3 = v_{N-N_2+1}.$$

Based on this construction, it is enough to begin with the network of three vertices q_1, q_2, q_3 and three resistances R_{q_i,q_j} between them. We transform this network into another network of four vertices and three resistances r_1, r_2, r_3 by the well known basic technique from elementary circuit theory known as the Δ-Y transform given by

$$r_j = \frac{R_{q_j,q_{j+1}} R_{q_j,q_{j-1}}}{R_{q_1,q_2} + R_{q_1,q_3} + R_{q_2,q_3}}, \tag{2.1}$$

and illustrated in Figure 5.

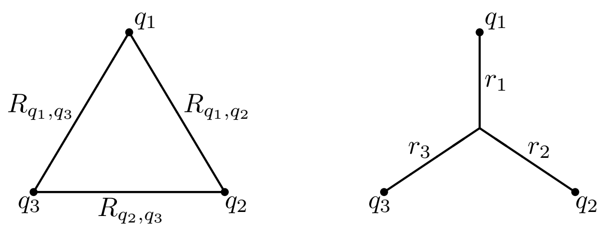

Figure 5. Illustration for the Δ-Y transform (2.1).

As the first consequence of these transformations we obtain the following lemma.

Lemma 2.1. *For each $k = 1, ..., N-1$ we have $R_k = R_{N-k}$.*

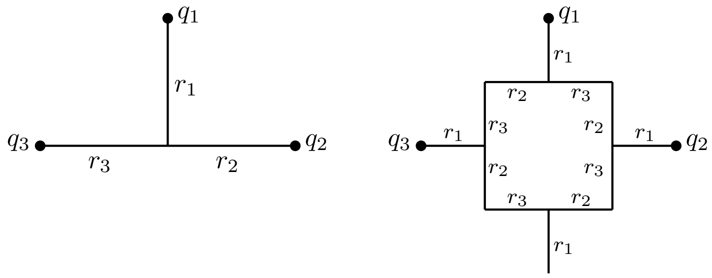

Figure 6. The transformed 0-depth and 1-depth networks in the case of the fractal cut square.

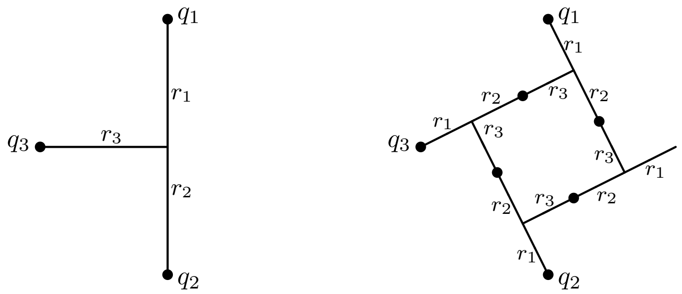

Figure 7. The transformed 0-depth and 1-depth networks in the case of the fractal red cross.

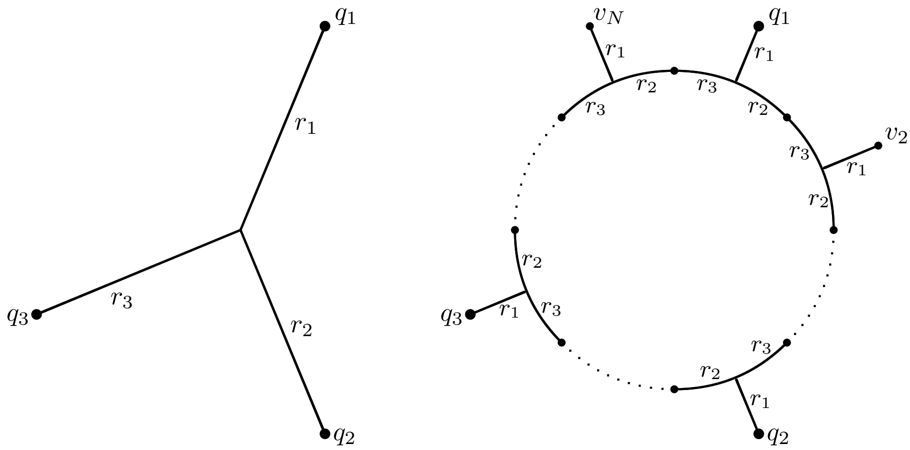

Figure 8. The transformed 0-depth and 1-depth networks in the case of large N.

Note that these symmetries of the resistances do not correspond to any symmetries of the N-gasket if $N_2 \neq N_3$. Also note that $r_2 \neq r_3$ if $N_2 \neq N_3$.

Proof of Lemma 2.1. One can see that in Figure 8 (see also Figures Figures 6 and 7 as examples) the resistances r_2 and r_3 are connected in pairs to form a resistances $r_2 + r_3$. Therefore in the network shown in Figure 8 on the right hand side, the resistance between v_i and v_{i+k} is the same as the resistance between v_i and v_{i-k}. This implies $R_k = R_{N-k}$ for $k = 1, ..., N-1$. □

3. Computations in a transformed N-gasket

Theorem 1. *Initial resistances r_1, r_2, r_3 correspond to a self-similar energy form on the (N_1, N_2, N_3)-gasket if and only if there is a constant c such that the following three equations hold*

$$cr_1 = r_1 + \frac{N_2 N_3}{N}(r_2 + r_3) \tag{3.1}$$

$$cr_2 = r_1 + \frac{N_1 N_3}{N}(r_2 + r_3) \tag{3.2}$$

$$cr_3 = r_1 + \frac{N_1 N_2}{N}(r_2 + r_3) \tag{3.3}$$

Proof. The network in Figure 8 is the same as the network in Figure 9 if we are

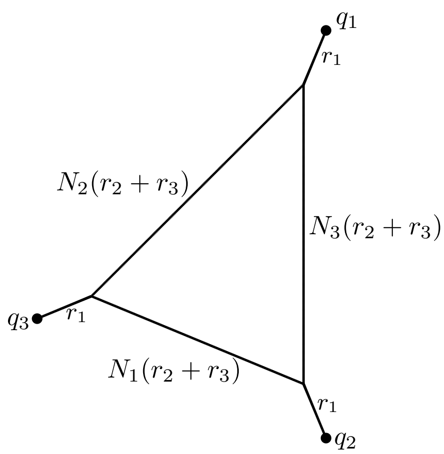

Figure 9. Illustration for the proof of Theorem 1.

concerned with the effective resistances between the three points q_1, q_2, q_3. In the latter network we do the Δ-Y transform (2.1) with the triangle of resistances $N_3(r_2 + r_3)$, $N_2(r_2 + r_3)$ and $N_1(r_2 + r_3)$. Thus, the result follows from (1.1) and (2.1). □

Theorem 2. *Let $s = N + N_1 N_3 + N_1 N_2$. Then,*

$$c = \frac{s + \sqrt{s^2 + (2N_2 N_3 - N_1 N_3 - N_1 N_2) 4N}}{2N}. \tag{3.4}$$

Proof. By (3.1) we have

$$r_1 = \frac{1}{(c-1)} \frac{N_2 N_3 (r_2 + r_3)}{N},$$

which we then plug into (3.2) and (3.3) and obtain

$$cr_2 = \frac{1}{(c-1)} \frac{N_2 N_3 (r_2 + r_3)}{N} + \frac{N_1 N_3 (r_2 + r_3)}{N},$$

$$cr_3 = \frac{1}{(c-1)} \frac{N_2 N_3 (r_2 + r_3)}{N} + \frac{N_1 N_2 (r_2 + r_3)}{N}.$$

Then by adding these two lines and dividing by $r_2 + r_3$ we obtain the following quadratic equation for c

$$Nc^2 - (N + N_1N_3 + N_1N_2)c + N_1N_3 + N_1N_2 - 2N_2N_3 = 0.$$

This implies the result since it is easy to see that $c > 1$, which also follows from the general theory [18]. □

This result is simplified in the following situation.

Corollary 3.1. *If $N_1 = N_2 = N_3 = n$ then*

$$c = 1 + \frac{2}{3}n.$$

The case $n = 1$ corresponds to the Sierpiński gasket, the case $n = 2$ corresponds to the hexagasket, see Example 3.4 and Figure 1 . The larger values of n do not correspond to self-similar fractals in $\mathbb{R}^2$, but any (n, n, n)-can be constructed as an abstract metric space.

Now that we have solved for c, we can find values for r_1, r_2, r_3. Note that the resistances are defined up to a scalar multiplier, and so we have to choose a normalization. The simplest answer is obtained with the normalization $r_1 = 1$, and also we obtain the formulas with normalizations $r_2 + r_3 = 1$ and $r_1 + r_2 + r_3 = 1$ as corollaries.

Theorem 3. *For any normalization we have*

$$r_2 = \frac{r_1}{c}\Big(1 + (c-1)\frac{N_1}{N_2}\Big) \tag{3.5}$$

$$r_3 = \frac{r_1}{c}\Big(1 + (c-1)\frac{N_1}{N_3}\Big) \tag{3.6}$$

where c is given by (3.4).

Proof. Using equation (3.1) we obtain that $r_2 + r_3 = \frac{N(c-1)}{N_2N_3}r_1$, and so

$$cr_2 = r_1 + \frac{(c-1)}{N_1N_2}r_1,$$
$$cr_3 = r_1 + \frac{N(c-1)}{N_1N_3}r_1$$

from (3.2) and (3.3). □

Corollary 3.2. *When $r_2 + r_3 = 1$ we have*

$$r_1 = \frac{N_2N_3}{N(c-1)} \tag{3.7}$$

$$r_2 = \frac{N_2N_3 + N_1N_3(c-1)}{Nc(c-1)} \tag{3.8}$$

$$r_3 = \frac{N_2N_3 + N_1N_2(c-1)}{Nc(c-1)} \tag{3.9}$$

where c is given by (3.4).

Proof. Using equation (3.1) and $r_2 + r_3 = 1$ we obtain that

$$cr_1 = r_1 + \frac{N_2N_3}{N},$$

which imply (3.7). Then (3.8) and (3.9) follows from (3.2) and (3.3), again using the normalization $r_2 + r_3 = 1$. □

Corollary 3.3. *When* $r_1 + r_2 + r_3 = 1$ *we have*

$$r_1 = \frac{N_2 N_3}{N(c-1) + N_2 N_3}$$

where c *is given by (3.4), and* r_2, r_3 *can be computed by (3.5) and (3.6).*

Proof. If $r_2 + r_3 = 1 - r_1$ then (3.1) immediately imply the result. □

Example 3.4. *For the Sierpiński gasket in Figure 1, which is a* $(1,1,1)$*-gasket, we have*

$$c = \frac{5}{3}$$

and $r_1 = r_2 = r_3$.

For the pentagasket in Figure 1, which is a $(1,2,2)$*-gasket, we have*

$$c = \frac{9 + \sqrt{161}}{10}$$

and $r_2 = r_3 = \frac{\sqrt{161}-1}{16} r_1$.

For the hexagasket in Figure 1, which is a $(2,2,2)$*-gasket, we have*

$$c = \frac{7}{3}$$

and $r_1 = r_2 = r_3$.

For the fractal cut square in Figure 2, which is a $(2,1,1)$*-gasket, we have*

$$c = \frac{2 + \sqrt{2}}{2}$$

and $r_2 = r_3 = \sqrt{2} r_1$.

For the fractal red cross in Figure 2, which is a $(1,1,2)$*-gasket, we have*

$$c = \frac{7 + \sqrt{65}}{8}$$

and $r_2 = r_1$, $r_3 = \frac{2\sqrt{65}-5}{4} r_1$.

4. Resistances of the N-gasket using matrix computations

Having computed the resistance scaling factor c as well the resistances r_1, r_2, r_3, we are now interested in the resistances R_m between points v_k and v_{k+m}. Note that the Δ-Y transform allows to compute effective resistances between points v_k and v_{k+m}, but does not allow to compute R_m. Instead we use a method when we first compute certain harmonic functions, and then the resistances (see Appendix).

Notation 4.1. *Let* $j_1, ..., j_N$ *be the junction points between resistances* r_1, r_2, r_3*, as indicated in Figure 10. In particular, each point* j_k *is connected to the boundary point* v_k *by a resistance* r_1*. Let* h *be a harmonic function with boundary values* $h(v_1) = 1$ *and* $h(v_k) = 0$ *for* $k = 2, ..., N$*. Then we denote* $J_k = h(j_k)$ *for each* $k = 1, ..., N$*.*

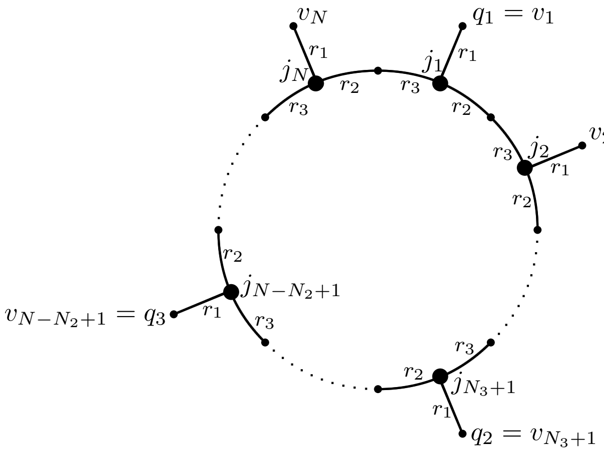

FIGURE 10. Notation 4.1 for the junction points in the transformed 1-depth network.

Lemma 4.2. *If $k \neq 1$ then*

$$J_k \frac{1}{r_1} + ((J_k - J_{k+1}) + (J_k - J_{k+1}))\frac{1}{r_2 + r_3} = 0$$

and

$$(J_1 - 1)\frac{1}{r_1} + (J_1 - J_2)\frac{1}{r_2 + r_3} + (J_1 - J_N)\frac{1}{r_2 + r_3} = 0$$

Proof. These are the equations of a harmonic function. □

We then rewrite the above equations as follows.

Lemma 4.3. *Let*

$$z = 2 + \frac{r_2 + r_3}{r_1}, \tag{4.1}$$

where

$$\frac{r_2 + r_3}{r_1} = \frac{1}{c}\Big(1 + (c-1)N_1\frac{N_2 + N_3}{N_2 N_3}\Big) \tag{4.2}$$

and c is given by (3.4). Then

$$J_N = J_2 = -J_2 + zJ_1 - \frac{r_2 + r_3}{r_1} \tag{4.3}$$

and if $k = 0, ..., N-1$

$$\begin{bmatrix} J_{k+1} \\ J_{k+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & z \end{bmatrix}^k \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \tag{4.4}$$

Proof. These are just transformed equations of the previous lemma. Note that $J_2 = J_N$ since the harmonic function is unique and the network in Figure 10 is symmetric around j_1. More exactly, resistances r_2 and r_3 are not symmetrically placed, but in this set-up we are interested only in the joint resistance $r_2 + r_3$ between each pair of points j_k and j_{k+1} (see also Lemma 2.1). Note that J_{N+1} is the same as J_1 by Notation 1.1. □

Next, we will find the powers of the matrix,

$$M = \begin{bmatrix} 0 & 1 \\ -1 & z \end{bmatrix} \tag{4.5}$$

Lemma 4.4.

$$M^k = \frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} \lambda_-^{k-1} - \lambda_+^{k-1} & \lambda_+^k - \lambda_-^k \\ \lambda_-^k - \lambda_+^k & \lambda_+^{k+1} - \lambda_-^{k+1} \end{bmatrix} \tag{4.6}$$

where

$$\lambda_+ = \frac{z + \sqrt{z^2 - 4}}{2} \tag{4.7}$$

$$\lambda_- = \frac{z - \sqrt{z^2 - 4}}{2} \tag{4.8}$$

are the eigenvalues of M.

Note that both λ_+ and λ_- are real, since we know that $z > 2$.

Proof. First, we find the eigenvectors of the matrix M, that is we solve for $MV_+ = \lambda_+ V_+$ and $MV_- = \lambda_- V_-$. Then we can find $M^k = CD^kC^{-1}$ where

$$D = \begin{bmatrix} \lambda_+^k & 0 \\ 0 & \lambda_-^k \end{bmatrix} \tag{4.9}$$

and $C = [V_+, V_-]$ is the matrix made from the eigenvectors of the matrix M. The matrix C is not unique and the easiest way to write is

$$C = \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \tag{4.10}$$

and

$$C^{-1} = \frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} -\lambda_- & 1 \\ \lambda_+ & -1 \end{bmatrix}. \tag{4.11}$$

Thus,

$$M^k = \frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{bmatrix} \begin{bmatrix} \lambda_+^k & 0 \\ 0 & \lambda_-^k \end{bmatrix} \begin{bmatrix} -\lambda_- & 1 \\ \lambda_+ & -1 \end{bmatrix}$$

Simplifying this equation, we get

$$M^k = \frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} \lambda_+^k & \lambda_-^k \\ \lambda_+^{k+1} & \lambda_-^{k+1} \end{bmatrix} \begin{bmatrix} -\lambda_- & 1 \\ \lambda_+ & -1 \end{bmatrix} =$$

$$\frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} -\lambda_+^k \lambda_- + \lambda_+ \lambda_-^k & \lambda_+^k - \lambda_-^k \\ -\lambda_+^{k+1} \lambda_- + \lambda_+ \lambda_-^{k+1} & \lambda_+^{k+1} - \lambda_-^{k+1} \end{bmatrix}$$

Notice that $\lambda_+ \lambda_- = 1$. Therefore, the lemma is proved. □

From the previous two lemmas we obtain the following corollary. Note that by our notation J_{N+1} is J_1.

Corollary 4.5. *For $k = 2, ..., N + 1$*

$$J_k = \frac{1}{\lambda_+ - \lambda_-} \left((\lambda_-^{k-2} - \lambda_+^{k-2}) J_1 + (\lambda_+^{k-1} - \lambda_-^{k-1}) J_2 \right) \tag{4.12}$$

Theorem 4.

$$J_1 = \left(\frac{r_2+r_3}{r_1}\right)\left(\frac{\lambda_+^N - \lambda_-^N}{2(\lambda_-^{N-1} - \lambda_+^{N-1}) + (\lambda_+^N - \lambda_-^N)z - 2(\lambda_+ - \lambda_-)}\right) \tag{4.13}$$

and

$$J_2 = \frac{1}{2}\left(zJ_1 - \frac{r_2+r_3}{r_1}\right) \tag{4.14}$$

where z, $(r_2+r_3)/r_1$, λ_+, λ_- are given by formulas (4.1), (4.2), (4.7), (4.8) respectively.

Proof. Equation (4.14) follows from (4.3). Then given that $M^{N-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we have

$$\begin{bmatrix} J_N \\ J_{N+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \end{bmatrix}$$

However, we know that $J_{N+1} = J_1$. So after matrix multiplication we get that $J_1 = CJ_1 + DJ_2$. Substitution of J_2 gives us that

$$J_1 = CJ_1 + \frac{1}{2}D(zJ_1 - \frac{r_2+r_3}{r_1})$$

which implies

$$J_1 = \frac{r_2+r_3}{r_1}\frac{D}{2C+zD-2}.$$

Then (4.13) follows from (4.6). □

Remark 4.6. *Note that the values J_k are important since they allow to compute so called harmonic matrices, that is transformations $h \mapsto h \circ \psi_i$ acting on the space of harmonic functions* [18, 38, 40].

Recall that, by definition, R_m is the resistances between the boundary points v_1 and v_{m+1} in the N-gasket. The set of these resistances is defined uniquely up to a constant multiplier. It seems the most convenient to normalize these resistances so that $R_1 = 1$.

Lemma 4.7. *Up to a multiplicative constant, which depends on the normalization, we have*

$$R_m = \frac{r_1}{J_{m+1}} \tag{4.15}$$

where J_{m+1} is given by (4.12).

Proof. According to Notation 4.1, the harmonic function h that is considered in this section is the same as a harmonic function h_y considered in the Appendix if $y = v_1$. Moreover, by definition R_m is the resistance between v_1 and v_{m+1} in the network which is the trace of of the network in Figure 10 on the boundary $\{v_1, ..., v_N\}$. Therefore $\frac{1}{R_m} = C'_{v_1,v_{m+1}}$ in the notation of Theorem 10. Therefore, by Theorem 10 with $x = v_{m+1}$, we have that

$$\frac{1}{R_m} = C_{v_{m+1},j_{m+1}}h(j_{m+1}) = \frac{1}{r_1}J_{m+1}.$$

Note that in this case the sum in the second formula in Theorem 10 has only one nonzero term with $p = j_{m+1}$. □

Theorem 5. *If the normalization of resistances is such that $R_1 = 1$, then*

$$R_m = \frac{\lambda_+ - \lambda_-}{\lambda_+^m - \lambda_-^m + \frac{J_1}{J_2}\left(\lambda_-^{m-1} - \lambda_+^{m-1}\right)} \tag{4.16}$$

where λ_+, λ_-, J_1, J_2 are given by formulas (4.7), (4.8), (4.13), (4.14) respectively.

Proof. According to Lemma 4.7, we have

$$R_m = const\frac{r_1}{J_{m+1}} \tag{4.17}$$

where J_{m+1} is given by (4.12) and *const* is a normalization constant. We choose this constant by the condition $R_1 = 1$. □

5. Asymptotic behavior

In this section we study the asymptotic of the resistances R_m and the resistance scaling factor c as N_1, N_2, N_3 tend to infinity. More precisely, we assume that $N_1(n), N_2(n), N_3(n)$ depend on a parameter n in a given way and consider various asymptotics as $n \to \infty$. Our main result in this section is the following theorem.

Theorem 6. *Assume that*

$$N_1(n) = \alpha n + o(n)_{n\to\infty},$$
$$N_2(n) = \beta n + o(n)_{n\to\infty},$$
$$N_3(n) = \gamma n + o(n)_{n\to\infty},$$

where α, β, γ are positive constants. Then we have the following limits

$$\lim_{n\to\infty} \frac{c(n)}{n} = \frac{\alpha(\beta+\gamma)}{\alpha+\beta+\gamma}, \tag{5.1}$$

and

$$\lim_{n\to\infty} R_m(n) = R_{m,\infty} = \lambda_{+,\infty}^{|m|-1} \tag{5.2}$$

where

$$\lambda_{+,\infty} = \frac{z_\infty + \sqrt{z_\infty^2 - 4}}{2} \tag{5.3}$$

and

$$z_\infty = 2 + \alpha\frac{\beta+\gamma}{\beta\gamma}. \tag{5.4}$$

Note that the set of resistances $\{R_m(n)\}$ is defined up to a scalar multiplier, and for convenience we use the normalization with $R_1(n) = 1$.

Note also that according to our notation $R_{-1} = R_{N-1}$, $R_{-2} = R_{N-2}$ etc.

Proof. Formula (5.1) follows from (3.4) since

$$N(n) = (\alpha+\beta+\gamma)n + o(n)_{n\to\infty},$$

and

$$s(n) = \alpha\gamma n^2 + \alpha\beta n^2 + o(n^2)_{n\to\infty}.$$

Then from (3.7) and (4.1) we obtain the limits

$$\lim_{n\to\infty} r_1(n) = \frac{\beta\gamma}{\alpha(\beta+\gamma)} = r_{1,\infty}, \tag{5.5}$$

given the normalization $r_2 + r_3 = 1$, and

$$\lim_{n\to\infty} z(n) = \frac{\alpha\beta + 2\beta\gamma + \alpha\gamma}{\beta\gamma} = z_\infty. \tag{5.6}$$

Hence we also have the limits

$$\lim_{n\to\infty} \lambda_+(n) = \frac{z_\infty + \sqrt{z_\infty^2 - 4}}{2} = \lambda_{+,\infty} \tag{5.7}$$

and

$$\lim_{n\to\infty} \lambda_-(n) = \frac{z_\infty - \sqrt{z_\infty^2 - 4}}{2} = \lambda_{-,\infty} \tag{5.8}$$

Moreover,

$$\lim_{n\to\infty} J_1(n) = \frac{\alpha(\beta+\gamma)}{\beta\gamma\sqrt{z_\infty^2 - 4}} = J_{1,\infty} \tag{5.9}$$

and

$$\lim_{n\to\infty} J_2(n) = \frac{\alpha(\beta+\gamma)}{2\beta\gamma}\left(\frac{z}{\sqrt{z_\infty^2-4}} - 1\right) = \lambda_{-,\infty} J_{1,\infty} = J_{2,\infty} \tag{5.10}$$

which imply

$$\lim_{n\to\infty} J_k(n) = \lambda_{-,\infty}^{|k-1|} J_{1,\infty} \tag{5.11}$$

for $k \geqslant 0$; note that according to our notation $J_0 = J_N$, $J_{-1} = J_{N-1}$ etc. Then (5.2) follows from (4.17). □

Remark 5.1. *Note that we use a specific normalization for the resistances. With the normalization $r_2 + r_3 = 1$ we have*

$$\lim_{n\to\infty} R_m(n) = \frac{\beta^2\gamma^2\sqrt{z_\infty^2-4}}{\alpha^2(\beta+\gamma)^2}\lambda_{+,\infty}^{|m|} = \lambda_{+,\infty}^{|m|-1} R_{1,\infty} \tag{5.12}$$

for $m > 0$. Formula

$$\lim_{n\to\infty} R_m(n) = \lambda_{+,\infty}^{|m|-1} R_{1,\infty} \tag{5.13}$$

is also true for normalizations $r_1 = 1$ and $r_1 + r_2 + r_3 = 1$.

Corollary 5.2. *Assume that $N_1 = N_2 = N_3 = n \to \infty$. Then we have the following limits*

$$\lim_{n\to\infty} \frac{c(n)}{n} = \frac{2}{3}$$

and

$$\lim_{n\to\infty} R_m(n) = \left(2+\sqrt{3}\right)^{|m|-1}$$

with normalization $R_1 = 1$.

Corollary 5.3. *Assume that the N-gasket F is a polygasket. Then we have the following limits*

$$\lim_{n\to\infty} \frac{c(n)}{n} = 1$$

and

$$\lim_{n\to\infty} R_m(n) = \left(3+2\sqrt{3}\right)^{|m|-1}$$

with normalization $R_1 = 1$.

Proof. It is easy to obtain from elementary geometry, see [33, 36, 38, 41], that an N-gasket is a polygasket if and only if N is not divisible by 4 and $N_2 = N_3 = [N/4]+1$, where $[\cdot]$ means the integer part of a number. □

Theorem 7. *Assume that*

$$N_1(n) = \alpha,$$
$$N_2(n) = \beta n + o(n)_{n\to\infty},$$
$$N_3(n) = \gamma n + o(n)_{n\to\infty},$$

where α, β, γ are positive constants. Then we have the following limits

$$\lim_{n\to\infty} \frac{c(n)}{\sqrt{n}} = \sqrt{2\frac{\beta\gamma}{\beta+\gamma}}, \tag{5.14}$$

and

$$\lim_{n\to\infty} \frac{R_m(n)}{R_k(n)} = 1 \tag{5.15}$$

for any nonzero n, k.

Proof. Formula (5.14) follows from (3.4) since

$$N(n) = (\beta+\gamma)n + o(n)_{n\to\infty},$$

and

$$s(n) = (\alpha+1)(\beta+\gamma)n + o(n)_{n\to\infty}.$$

Then from (4.2) we obtain

$$\lim_{n\to\infty} \frac{r_2(n)+r_3(n)}{r_1(n)} = 0 \tag{5.16}$$

which implies

$$\lim_{n\to\infty} z(n) = 2. \tag{5.17}$$

Hence we also have the limits

$$\lim_{n\to\infty} \lambda_+(n) = \lim_{n\to\infty} \lambda_-(n) = 1. \tag{5.18}$$

This implies (5.15). □

Theorem 8. *Assume that*

$$N_1(n) = \alpha n + o(n)_{n\to\infty},$$
$$N_2(n) = \beta n + o(n)_{n\to\infty},$$
$$N_3(n) = \gamma,$$

where α, β, γ are positive constants. Then we have the following limits

$$\lim_{n\to\infty} \frac{c(n)}{n} = \frac{\alpha\beta}{\alpha+\beta}, \tag{5.19}$$

and

$$\lim_{n\to\infty} \frac{R_m(n)}{R_k(n)} = 0 \tag{5.20}$$

if $|m| > |k| > 0$.

Proof. Formula (5.19) follows from (3.4) since

$$N(n) = (\alpha + \beta)n + o(n)_{n\to\infty},$$

and

$$s(n) = \alpha\beta n^2 + o(n^2)_{n\to\infty}.$$

Then from (4.2) we obtain

$$\frac{r_2(n) + r_3(n)}{r_1(n)} = \frac{\alpha}{\gamma} n + o(n)_{n\to\infty} \tag{5.21}$$

which imply (5.20). □

Note that case

$$N_1(n) = \alpha n + o(n)_{n\to\infty},$$
$$N_2(n) = \beta,$$
$$N_3(n) = \gamma n + o(n)_{n\to\infty}$$

can be treated analogously.

Theorem 9. *Assume that*

$$N_1(n) = \alpha n + o(n)_{n\to\infty},$$
$$N_2(n) = \beta,$$
$$N_3(n) = \gamma,$$

where α, β, γ *are positive constants. Then we have the following limits*

$$\lim_{n\to\infty} c(n) = \beta + \gamma, \tag{5.22}$$

and

$$\lim_{n\to\infty} \frac{R_m(n)}{R_k(n)} = 0 \tag{5.23}$$

if $|m| > |k| > 0$.

Proof. Formula (5.22) follows from (3.4) since

$$N(n) = \alpha n + o(n)_{n\to\infty},$$

and

$$s(n) = (1 + \gamma + \beta)\alpha n + o(n)_{n\to\infty}.$$

Then from (4.2) we obtain

$$\lim_{n\to\infty} z(n) = \infty$$

which imply (5.23). □

Corollary 5.4. *For any value of the parameter* n *we denote the spectral dimension of the Laplacian on the* N*-gasket by* $d_S(n)$ *and the Hausdorff dimension of the* N*-gasket with respect to the effective resistance metric by* $d_H(n)$*, as in Remark 1.3. Then*

(1) *With the assumptions of Theorem 6 or 8 we have*

$$\lim_{n\to\infty} d_H(n) = d_S(n) = 1; \tag{5.24}$$

(2) *With the assumptions of Theorem 7 we have*

$$\lim_{n\to\infty} d_H(n) = 2 \qquad \textit{and} \qquad \lim_{n\to\infty} d_S(n) = \frac{4}{3};$$

(3) *With the assumptions of Theorem 9 we have*

$$\lim_{n\to\infty} d_H(n) = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} d_S(n) = 2.$$

Note that formula (5.24) applies, in particular, to polygaskets with the number of sides in the corresponding polygon approaching infinity.

6. Appendix: resistor networks, harmonic functions, traces of Dirichlet forms and random walks

In this section we recall some basic facts about laws of resistor networks (electrical circuits) and their relation to the finite dimensional Dirichlet forms and to random walks. The timeless classical reference on the relation between random walks and discrete potential theory is [9]. A newer and more relevant reference with respect to the electrical circuits formalism is [8]. As an expository remark we note there exists a different use of probabilistic techniques in analysis on fractals which uses notions of Martin and Poisson boundaries [6, 7, and references therein].

For a single resistor the *Ohm's Law* is that *the voltage V is equal to the current I times the resistance R,* which is expressed by the formula

$$V = IR.$$

In addition, we have the formula for the electrical energy

$$E = VI.$$

If we define a conductance C by $C = \frac{1}{R}$, then we have

$$E = CV^2. \tag{6.1}$$

In a resistor network we also have the *Voltage Law: the total change of voltage over a closed loop is zero*, and the *Current Law: at an inner junction point, the same amount of current flows in as out.*

These laws imply two basic rules for resistances. If there are two consecutively connected resistances R_1 and R_2, also called series resistors, then the total resistance is $R_1 + R_2$. If there are two parallel resistances R_1 and R_2, then the total resistance is $\frac{R_1 R_2}{R_1+R_2}$. So if there are two parallel conductances C_1 and C_2, then the total conductances is $C_1 + C_2$. Another rule is the Δ-Y transform (2.1).

In a network we denote $V(x, y)$ the voltage change from x to y. Then the Voltage Law implies that there is a function $U(x)$, called electric potential, such that $V(x, y) = U(x) - U(y)$ for any two points x, y in the network. The potential is unique up to an additive constant. An easy construction of a potential is as follows. Assign some potential U_0 to some x_0. Then for another x consider a path $x_0, x_1, ..., x_n$ such that x_j and x_{j+1} are connected. Then define

$$U(x) = U_0 + V(x_0, x_1) + V(x_1, x_2) + \cdots + V(x_{n-1}, x).$$

The Voltage Law implies that $U(x)$ is the same for different paths from x_0 to x.

The conductance between any two junction points x and y in the network is denoted by $C_{x,y} \geqslant 0$. The conductance $C_{x,y}$ is zero by definition if the points x

and y are not directly connected. For a given function f defined on junction points of the network we define its energy by

$$\mathcal{E}(f,f) = \sum_{x,y} C_{x,y}(f(x) - f(y))^2.$$

Note that, according to (6.1), if $f = U$ is a potential, then the energy $\mathcal{E}$ is the same as the total electrical energy, which is the sum of energies (6.1) over all resistors in the network.

From now on we will denote the set of junction points in the network by V, which should not be confused with the voltage mentioned in the beginning of this section. Also, from now on the junction points will be called vertices.

Definition 6.1. *A function h is called harmonic at p if*

$$\sum_{q \in V} C_{p,q}(h(p) - h(q)) = 0.$$

We call h a harmonic function if it is harmonic at every inner vertex.

It is an easy consequence of Ohm's and Current Laws that any electric potential function $U(x)$ is a harmonic function.

We always assume that a network is connected in the sense that for any vertices x and y there is a path $x = x_0, x_1, ..., x_n = y$ such that each pair x_j and x_{j+1} is connected by a positive conductance.

It is easy to see from the connectedness of the network, definition of a harmonic function and the positivity of conductances that the *Strong Maximum Principle* holds for any harmonic function: *a harmonic function can not have a local maximum or minimum at an inner vertex.* From this principle one can obtain another important fact about harmonic functions: *for any set of boundary values there exists a unique harmonic function with these values.* In other terms it means the following. Let V_{inner} denote the set of inner vertices and ∂V denote its complement in V. Naturally ∂V is called the boundary. Then *the unique harmonic extention property* is that for any function g on the boundary ∂V there is a unique harmonic function h such that $g = h\Big|_{\partial V}$. To prove the unique harmonic extention property notice that the equations in the definition of a harmonic function are linear, the number of equations is the same as the number of variables, and the only solution of the homogeneous system is zero by the Strong Maximum Principle.

The formula for the harmonic extention can be written as follows. For each $y \in \partial V$ we define by $h_y(x)$ the unique harmonic function such that $h_y(y) = 1$ and $h_y(x) = 0$ for every other $x \in \partial V$. Then the harmonic extention formula is

$$h(x) = \sum_{y \in \partial V} h(y) h_y(x)$$

for any $x \in V$.

The last of the principles is the *Minimal Energy Principle* that is *the harmonic function has the minimal energy among the functions with given boundary values.* This principle is proved directly by minimization in the definition of the energy. The Minimal Energy Principle implies that the *Effective conductance between p and q* is

$$C_{\text{effect}}(p,q) = \frac{E(h,h)}{(h(p) - h(q))^2},$$

where h is any non constant harmonic function in the network with the boundary values given at $\partial V = \{p, q\}$. The effective resistance is the reciprocal of the effective conductance. Note that here and in other situations the notion of the boundary ∂V of a finite set V is flexible, and can depend on a particular function and on other considerations.

The trace of the resistance form $\mathcal{E}$ on the boundary ∂V is defined as the unique resistance form $Tr_{\partial V}\mathcal{E}$ such that for any function g on ∂V we have

$$Tr_{\partial V}\mathcal{E}(g, g) = \mathcal{E}(h, h),$$

where h is the unique harmonic extention of g. The next theorem is the main result of this section.

Theorem 10.

$$Tr_{\partial V}\mathcal{E}(g, g) = \sum_{x,y\in\partial V} C'_{x,y}(g(x) - g(y))^2,$$

where

$$C'_{x,y} = \sum_{p\in V} h_y(p)C_{x,p}.$$

Note that the strong maximum principle implies that $h_y(p) \geqslant 0$ and so $C'_{x,y} \geqslant 0$.

Proof. For any function h we have

$$\begin{aligned}\mathcal{E}(h, h) &= \sum_{x,p\in V} C_{x,p}(h(x) - h(p))^2 \\ &= \sum_{x,p\in V} C_{x,p}((h(x)(h(x) - h(p)) + h(p)(h(p) - h(x))) \\ &= 2\sum_{x,p\in V} h(x)\, C_{x,p}(h(x) - h(p))\end{aligned}$$

Let h be the unique harmonic extention to V of a function g defined on ∂V. Then $\sum_{p\in V} C_{x,p}(h(x) - h(p)) = 0$ if x is not a boundary point. Therefore

$$\mathcal{E}(h, h) = 2\sum_{x\in\partial V, p\in V} h(x)\, C_{x,p}(h(x) - h(p)).$$

For any $p \in V$ we have

$$\sum_{y\in\partial V} h_y(p) = 1.$$

This formula follows from the fact that the function which is identically one on ∂V has a unique harmonic extension which is identically one on V. Using this and the formula $h(x) = \sum_{y\in\partial V} h(y)h_y(x)$ we obtain

$$\begin{aligned}
\mathcal{E}(h,h) &= 2\sum_{x\in\partial V, p\in V} h(x)\, C_{x,p} \sum_{y\in\partial V} h_y(p)\,(h(x)-h(y)) \\
&= 2\sum_{x,y\in\partial V} h(x)\ (h(x)-h(y)) \sum_{p\in V} C_{x,p} h_y(p) \\
&= \sum_{x,y\in\partial V} \Big(h(x)\ (h(x)-h(y)) + h(y)\ (h(y)-h(x)) \Big) \sum_{p\in V} C_{x,p} h_y(p) \\
&= \sum_{x,y\in\partial V} (g(x)-g(y))^2 \sum_{p\in V} C_{x,p} h_y(p),
\end{aligned}$$

which completes the proof. □

It is well known that Dirichlet forms, resistor networks and symmetric random walks lead to essentially the same harmonic analysis on a finite set V (see for instance [8]). It is clear that Dirichlet forms and resistor networks are in one-to-one correspondence. A random walk X_n, $n = 0, 1, 2, ...$, is defined in terms of its transition probabilities $p(x, y)$, $x, y \in V$. This means that $X_{n+1} = y$ with probability $p(x, y)$ given that $X_n = x$. In what follows we assume that, for $x \neq y$,

$$p(x,y) = \frac{C_{x,y}}{\sum_{z\in V} C_{x,z}}$$

In this situation we'll say that the resistor network and the random walk are equivalent. Any random walk defined in this way is called symmetric. The symmetric random walks are in one-to-one correspondence with Dirichlet forms up to a constant multiple.

We assume that $\mathcal{E}(f, f) = 0$ if and only if f is constant. This means that the random walk X_n can reach any point with positive probability in a finite number of steps. In terms of the resistor network, this means that the network does not have more than one connected components.

Let $P_x\{\cdot\}$ denotes the probability distribution of the random walk X_n which starts at $X_0 = x$, that is $P_x\{X_0 = x\} = 1$, and let E_x denotes the expectation with respect to $P_x\{\cdot\}$. Then a function h is harmonic at $x \in V$ if and only if $h(x) = E_x h(X_1) = \sum_{y\in V} p(x, y)h(y)$. Moreover, $h(x)$ can be expressed through its boundary values as follows. Let τ be the first time X_n reaches ∂V, that is $\tau = \min\{n : X_n \in \partial V\}$. Then $h(x) = E_x h(X_\tau)$ for any $x \in V$. Note that this expression depends only on the values of h on ∂V since $X_\tau \in \partial V$ with probability one. In particular, $h_y(x) = P_x\{X_\tau = y\}$.

Let $p'(x, y) = P_x\{X_{\tau'_x} = y\}$ where τ'_x be the first time X_n reaches $\partial V\backslash\{x\}$, that is $\tau'_x = \min\{n : X_n \in \partial V\backslash\{x\}\}$. In other words, the random walk X_n that starts at $x \in \partial V$ is killed at $\partial V\backslash\{x\}$. Then $p'(x, y)$ is the probability that this walk is killed at y. Then it is easy to see from the Markov property that $p'(x, y)$ are the transition probability of the random walk on ∂V that corresponds to the Dirichlet form $Tr_{\partial V}\mathcal{E}$. This is the relation which was used in [4, 20, and references therein] and other works to define self-similar diffusion on the fractals. One can obtain other relations between random walks and resistance forms. For instance, $c = \frac{1}{1-p''_x}$ where p''_x is the probability that the random walk X_n that starts at $x \in \partial V$ returns to x before it is killed when it hits $\partial V\backslash\{x\}$.

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