Analysis on Apollonian Gaskets

Sharat Chandra, Fiona Galzarano, Dr. Robert Strichartz

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Apollonian Gaskets
Given three mutually tangent circles $C_1, C_2, C_3$, Apollonious showed that there are exactly 2 choices of a 4th circle $C_4$ and $C'_4$ that are tangent to the first three.
Descartes Theorem:
If $C_1$ has curvature $k_1$, $C_2$ has curvature $k_2$, $C_3$ has curvature $k_3$, $C_4$ has curvature $k_4$, and $C'_4$ has curvature $k'_4$, then:

$$k_4 = k_1 + k_2 + k_3 + 2(k_1 k_2 + k_1 k_3 + k_2 k_3)$$

$$k'_4 = k_1 + k_2 + k_3 - 2(k_1 k_2 + k_1 k_3 + k_2 k_3)$$
$r_1 \geq r_2 \geq r_3$ which implies $l_1 \geq l_2 \geq l_3$
Calculating Arc Lengths

We can use law of cosines to calculate arc lengths:

\[
l_1 = r_1 \cos^{-1} \left( \frac{(r_1 + r_2)^2 + (r_1 + r_3)^2 - (r_2 + r_3)^2}{2(r_1 + r_2)(r_1 + r_3)} \right)
\]

\[
l_2 = r_2 \cos^{-1} \left( \frac{(r_1 + r_2)^2 + (r_2 + r_3)^2 - (r_1 + r_3)^2}{2(r_1 + r_2)(r_2 + r_3)} \right)
\]

\[
l_3 = r_3 \cos^{-1} \left( \frac{(r_1 + r_3)^2 + (r_2 + r_3)^2 - (r_1 + r_2)^2}{2(r_1 + r_3)(r_2 + r_3)} \right)
\]
For a function $u(x)$ evaluated along the nodes of a discrete approximation of an Apollonian Gasket $m$, we can define energy locally as:

$$
\varepsilon_m(u) = \sum_{j \neq k} \frac{|u(x_j) - u(x_k)|^2}{d(x_j, x_k)}
$$
A Delta network of resistors can be made into an equivalent Y configuration, where the new resistances are such that the effective resistance between any two original nodes is the same.
Delta-Y Transformations

The formulas for the transformation from a "Delta" configuration to a "Y" configuration are as follows:

\[ R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \]
\[ R_2 = \frac{R_a R_c}{R_a + R_b + R_c} \]
\[ R_3 = \frac{R_a R_b}{R_a + R_b + R_c} \]
Delta-Y Transformations

The formulas for the transformation from a "Y" configuration to a "Delta" configuration are as follows:

\[ R_a = \left( \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_1} \right) \]

\[ R_a = \left( \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_1} \right) \]

\[ R_a = \left( \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_3} \right) \]
Going from Stage $n + 1$ to Stage $n$ of a Gasket
Searching for Renormalization

- In the final stage, we may return the network into its Delta form and see the effective resistance of a single triangle.

- Notice that the resistances are scaled from their original values by \( \frac{5}{3} \). This is the renormalization constant for the Sierpinski Gasket.

- We wish to repeat this procedure of subdividing the Apollonian gasket \( m \) levels deep, and reconstructing the effective level 0 gasket.

- We hope that we may find another renormalization constant, perhaps strongly dependent on the initial configuration of the gasket.
Label the arc lengths such that $l_1 \geq l_2 \geq l_3$. Define $l'_2 = \frac{l_2}{l_1}$, $l'_3 = \frac{l_3}{l_1}$. This allows us to examine the proportions of arc lengths in different Apollonian Gasket constructions. To assist us in our analysis, we plotted graphs of $l'_2$ vs $l'_3$ for thousands of randomly generated initial radii in Matlab, with the following restrictions:

- $l_1 = 1$
- $l_2 \leq 1$
- $l_3 \leq l_2$
$l'_2$ vs $l'_3$

Stage 0
$l'_2$ vs $l'_3$

Stage 1
$l_2'$ vs $l_3'$

Stage 2
$l_2 \ vs \ l_3$

Stage 3
Measure

Measure is defined recursively:

\[ \mu(T_0) = A(T_0) \]

For \( j > 0 \):

\[ \mu(T_j) = \frac{A(T_j)}{A(T_{j-1}) - \pi r_{j-1}^2} \]

where \( r_{j-1} \) is the radius of the circle found from applying Descartes Theorem.

Measure has the property:

\[ \mu(T_{j-1}) = \mu(T_j) + \mu(T_j') + \mu(T_j'') \]
Types of Partitions:

- **Level Partition** - At each stage, add a center circle to each triangular region.

- **Measure Partition** - Fix $\varepsilon > 0$. Check each triangular region $T$. If $\mu(T) > \varepsilon$, add a center circle to break $T$ into three new triangular regions. Repeat this process for the new regions.

- **Area Partition** - Similar to a measure partition, but with area.

- **Size Partition** - Similar to a measure partition, but with size. Size is calculated recursively, by multiplying the length of the longest side of the triangular intersection by the size value at the previous level.
Example Threshold Partitions

Level Partition at Stage 3

Measure Partition with $\varepsilon = 0.005$
Our Definition:

$$-\Delta u(x) = \sum_{(y \sim x)} \frac{u(x) - u(y)}{d(x,y)\mu(x)}$$

The matrix is given by,

$$-\Delta u(x)_{a,a} = \sum_{y \sim a} \frac{1}{d(y,a)\mu(a)}$$

When \( z \neq a \),

$$-\Delta u(x)_{a,z} = \begin{cases} 
0 & \text{if } z \not\sim a \\
-\frac{1}{d(a,z)\mu(a)} & \text{if } z \sim a 
\end{cases}$$

Different Laplacians based on Partition Types:

- Level Partition
- Measure Partition
- Area Partition
- Size Partition
$$r_1 = 1, \quad r_2 = 0.7, \quad r_3 = 0.2$$
$r_1 = 1$, $r_2 = 0.7$, $r_3 = 0.2$

**Graphs:**

- **Top Left:** Number of eigenvalues less than $t$.
- **Top Right:** $\ln(N(t))$ vs. $t$ with slope $0.6118$.
- **Bottom:** $\ln(N(t)/t^{0.6118})$ vs. $\ln(t)$. 
Laplacians
Size Partition

\[ r_1 = 1, \quad r_2 = 0.7, \quad r_3 = 0.2 \]

![Graphs showing number of eigenvalues less than \( t \), \( \ln(N(t)) \), and \( \ln(N(t)/t^{0.5773}) \) against \( \ln(t) \). The slope of the line in the \( \ln(N(t)) \) vs. \( \ln(t) \) graph is 0.5773.](Image)
