Mathematics 2110 SOLUTIONS
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Wednesday, October 15, 2008

1. Let $f(x,y,z) = x^2y + y^3\sin(z^2)$. Find all three rst partial derivatives of f.

   Solution:

   $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2 + 3y^2\sin(z^2)$, $\frac{\partial f}{\partial z} = 2y^3z\cos(z^2)$.

2. Consider the function $z = g(x,y)$ de ned implicitly by the equation $x^2y + y^3\sin(z^2) = 1$ in the neighborhood of some point. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. *Extra Credit: Why did I add the description about a neighborhood of a point?*

   Solution:

   Letting $f(x,y,z) = x^2y + y^3\sin(z^2)$:

   $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{2xy}{2y^3z\cos(z^2)} = -\frac{x}{y^2z\cos(z^2)}$$

   $$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{x^2 + 3y^2\sin(z^2)}{2y^3z\cos(z^2)}$$

3. Let $f(x,y,z) = x^2y + y^3\sin(z^2)$. Find ∇f in general and also calculate $\nabla f\big|_{(-1,1,\sqrt{\pi})}$.

   Solution:

   $\nabla f = \langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle = \langle 2xy, x^2 + 3y^2\sin(z^2), 2y^3z\cos(z^2)\rangle$

   $\nabla f\big|_{(-1,1,\sqrt{\pi})} = \langle -2, 1, -2\sqrt{\pi}\rangle$, since $\sin\pi = 0$ and $\cos\pi = -1$.

4. Find an equation for the plane tangent to $x^2y + y^3\sin(z^2) = 1$ at $(-1, 1, \sqrt{\pi})$.

   Solution:

   $-2(x+1) + 1(y-1) - 2\sqrt{\pi}(z - \sqrt{\pi}) = 0$ or $2x + y - 2\sqrt{\pi}z + 1 + 2\pi = 0$.

5. Use the Chain Rule to evaluate $\frac{dw}{dt}$, where $w = x^2y + y^3\sin(z^2)$ with $x = 5t$, $y = t^2$ and $z = 4t + 9$. *Please do not try to simplify this.*

Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = 2xy \cdot 5 + (x^2 + 3y^2\sin(z^2)) \cdot 2t + 2y^3 z\cos(z^2) \cdot 4.$

6. Use the Chain Rule to evaluate $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$, where $w = x^2y + y^3\sin(z^2)$ with $x = s^2 + 5t$, $y = s^3t^2$ and $z = e^{st}$. *Please do not try to simplify this.*

Solution:

$\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = 2xy \cdot 2s + (x^2 + 3y^2\sin(z^2)) \cdot 3s^2t^2 + 2y^3z\cos(z^2) \cdot te^{st}.$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = 2xy \cdot 5 + (x^2 + 3y^2\sin(z^2)) \cdot 2s^3t + 2y^3z\cos(z^2) \cdot se^{st}.$

7. Let $f(x, y, z) = x^2y + y^3\sin(z^2)$ and let $\mathbf{u}$ be the unit vector in the direction of $\langle 5, -2, 9\rangle$. Find the directional derivative $D_{\mathbf{u}}f$.

Solution: $|\langle 5, -2, 9\rangle| = \sqrt{5^2 + (-2)^2 + 9^2} = \sqrt{110}$, so $\mathbf{u} = \frac{\langle 5, -2, 9\rangle}{\sqrt{110}}$,

$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle 2xy, x^2 + 3y^2\sin(z^2), 2y^3z\cos(z^2)\rangle \cdot \frac{\langle 5, -2, 9\rangle}{\sqrt{110}}$

$= \frac{1}{\sqrt{110}} \cdot [5(2xy) - 2(x^2 + 3y^2\sin(z^2)) + 9(2y^3z\cos(z^2))]$

8. Find all critical points of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. Examine each critical point and determine what you can about it in terms of whether it's a local minimum, local maximum or saddle point.

Solution:

$f_x = 4x^3 - 4x + 4y = 4(x^3 - x + y)$, $f_y = 4y^3 + 4x - 4y = 4(y^3 + x - y)$

$f_{xx} = 4(3x^2 - 1)$, $f_{xy} = 4$, $f_{yy} = 4(3y^2 - 1)$

The critical points occur where both rst partial derivatives are equal to 0, so we solve:
$x^3 - x + y = 0$, $y^3 + x - y = 0$
$(x^3 - x + y) + (y^3 + x - y) = 0 + 0$
$x^3 + y^3 = 0.$
Clearly, the only solution to the last equation is $y = -x$. Plugging into the rst equation, we get $x^3 - x + (-x) = 0$, $x^3 - 2x = 0$, $x(x^2 - 2) = 0$. This clearly has three solutions, $0, \sqrt{2}$ and $-\sqrt{2}$.

We thus have critical points $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$.

At $(0, 0)$, $f_{xy}^2 - f_{xx}f_{yy} = 4^2 - (-4)(-4) = 0$, so the Second Derivative Test is inconclusive.

However, we note that $f(x, 0) = x^4 - 2x^2 = x^2(x^2 - 2) < 0$ if x is close to 0, while $f(x, x) = x^4 + x^4 - xx^2 + 4x^2 - 2x^2 = 2x^4 > 0$ if x is close to 0, so $(0, 0)$ is a saddle point.

At both $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$, $f_{xy}^2 - f_{xx}f_{yy} = 4^2 - 20 \cdot 20 < 0$ and $f_{xx} = 20 > 0$, so there is a local minimum at both places.

9. Find all critical points of the function $f(x,y) = (x-1)(y-1)(x+y-1)$. Examine each critical point and determine what you can about it in terms of whether it's a local minimum, local maximum or saddle point.

Solution:

$f_x = (y-1)(2x+y-2)$, $f_y = (x-1)(x+2y-2)$

$f_{xx} = 2(y-1)$, $f_{xy} = 2x+2y-3$, $f_{yy} = 2(x-1)$.

The critical points occur where both rst partial derivatives are equal to 0, so we solve:
$(y-1)(2x+y-2) = 0$, $(x-1)(x+2y-2) = 0$

We have four cases, each of which is easily solved:

Case 1: $y-1=0$, $x-1=0$. This gives the critical point $(1,1)$.

Case 2: $y-1=0$, $x+2y-2=0$. Plugging $y=1$ into the second equation gives $x+2-2=0$, $x=0$, so we have the critical point $(0,1)$.

Case 3: $2x+y-2=0$, $x-1=0$. Plugging $x=1$ into the rst equation gives $2+y-2=0$, $y=0$, so we have the critical point $(1,0)$.

Case 4: $2x+y-2=0$, $x+2y-2=0$. Subtracting corresponding sides gives
$(2x+y-2)-(x+2y-2) = 0-0$, $x-y=0$, $x=y$. Plugging $y=x$ into the rst equation gives $2x+x-2=0$, $x=\frac{2}{3}$, so we have the critical point $(\frac{2}{3},\frac{2}{3})$.

At $(1,1)$, $f_{xy}^2 - f_{xx}f_{yy} = 1^2 - 0\cdot 0 > 0$, so this is a saddle point.

At $(0,1)$, $f_{xy}^2 - f_{xx}f_{yy} = (-1)^2 - (-2)(0) > 0$, so this is also a saddle point.

At $(1,0)$, $f_{xy}^2 - f_{xx}f_{yy} = (-1)^2 - (0)(-2) > 0$, so this is also a saddle point.

At $(\frac{2}{3},\frac{2}{3})$, $f_{xy}^2 - f_{xx}f_{yy} = (-\frac{1}{3})^2 - (-\frac{2}{3})(-\frac{2}{3}) < 0$, so there is a local extremum. Since $f_{xx} = -\frac{2}{3} < 0$, it is a local maximum.

10. Find the point of the plane $2x-3y-4z=25$ which is nearest to the point $(3,2,1)$. *Extra Credit: Do this two completely different ways. Even More Extra Credit: Do this three completely different ways.*

Solution:

We must minimize $f(x,y,z) = (x-3)^2 + (y-2)^2 + (z-1)^2$ subject to the constraint $g(x,y,z) = 2x-3y-4z = 25$. Using Lagrange Multipliers, we set $\nabla f = \lambda \nabla g$, $\langle 2(x-3), 2(y-2), 2(z-1)\rangle = \lambda\langle 2,-3,-4\rangle$. We thus solve:

$2(x-3) = 2\lambda$, $2(y-2) = -3\lambda$, $2(z-1) = -4\lambda$, $2x-3y-4z = 25$.

Solving for λ, we get $x = \lambda+3$, $y = -\frac{3}{2}\lambda+2$, $z = -2\lambda+1$, so
$2(\lambda+3) - 3(-\frac{3}{2}\lambda+2) - 4(-2\lambda+1) = 25$, $\frac{29}{2}\lambda - 4 = 25$, $\frac{29}{2}\lambda = 29$, $\lambda = 2$.

We thus get $x=5$, $y=-1$, $z=-3$.

The closest point is thus $(5,-1,-3)$.

11. Find the minimum value for $x^3 + y^3 + z^3$ among points on the plane $x + 4y + 9z = 28$ for which all the coordinates are positive.

Solution: If there is such a point, using Lagrange Multipliers it must satisfy $\nabla(x^3 + y^3 + z^3) = \lambda\nabla(x + 4y + 9z)$, or $\langle 3x^2, 3y^2, 3z^2\rangle = \lambda\langle 1, 4, 9\rangle$.

If we let $\lambda = 3\mu$, this simpli es to $\langle x^2, y^2, z^2\rangle = \mu\langle 1, 4, 9\rangle$, or

$x^2 = \mu$, $y^2 = 4\mu$, $z^2 = 9\mu$. Since all the values are positive, we have $x = \sqrt{\mu}$, $y = 2\sqrt{\mu}$, $z = 3\sqrt{\mu}$, so $\sqrt{\mu} + 4 \cdot 2\sqrt{\mu} + 9 \cdot 3\sqrt{\mu} = 28$, $36\sqrt{\mu} = 28$, $\sqrt{\mu} = \frac{28}{36} = \frac{7}{9}$.

We get the point $(\frac{7}{9}, \frac{14}{9}, \frac{21}{9})$.

It's not immediately obvious that this point yields a minimum rather than a maximum. The following is one way of showing it yields a minimum.

Any line in the plane through the point $(\frac{7}{9}, \frac{14}{9}, \frac{21}{9})$ has parametric equations $x = at + \frac{7}{9}$, $y = bt + \frac{14}{9}$, $z = ct + \frac{21}{9}$. Since any point on the line is on the plane $x + 4y + 9z = 28$, it follows that $(at + \frac{7}{9}) + 4(at + \frac{7}{9}) + 9(ct + \frac{21}{9}) = 28$, $(a + 4b + 9c)t = 0$, so $a = -(4b + 9c)$.

We thus have $x = -(4b + 9c)t + \frac{7}{9}$, $y = bt + \frac{14}{9}$, $z = ct + \frac{21}{9}$.

If we let $w = x^3 + y^3 + z^3$, then $w = [-(4b + 9c)t + \frac{7}{9}]^3 + [bt + \frac{14}{9}]^3 + [ct + \frac{21}{9}]^3$.

$w' = 3[(-(4b + 9c))[-(4b + 9c)t + \frac{7}{9}]^2 + b[bt + \frac{14}{9}]^2 + c[ct + \frac{21}{9}]^2]$.

$w'' = 2 \cdot 3[(4b + 9c)^2[-(4b + 9c)t + \frac{7}{9}] + b^2[bt + \frac{14}{9}] + c^2[ct + \frac{21}{9}]]$.

It's clear that $w'' > 0$ when $t = 0$, so w has a minimum when $t = 0$, i.e. at the point $(\frac{7}{9}, \frac{14}{9}, \frac{21}{9})$.

12. Use a double integral to nd the area of a unit circle. *You will need to represent the area as a double integral of some function over some plane region and then use an iterated integral to evaluate the double integral. It is likely that you'll need some of the integration techniques from Calculus II in order to evaluate the iterated integral.*

Solution: Area $= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y = 2\int_{-1}^{1} \sqrt{1 - x^2}\, dx$.

Using a trigonometric substitution with a right triangle with acute angle θ, opposite side x, adjacent side $\sqrt{1 - x^2}$, and hypotenuse 1.

Then $\sqrt{1 - x^2} = \cos\theta$, $x = \sin\theta$, $\dfrac{dx}{d\theta} = \cos\theta$, $dx = \cos\theta\, d\theta$, so $\int \sqrt{1 - x^2}\, dx = \int \cos\theta \cdot \cos\theta\, d\theta = \int \cos^2\theta\, d\theta = \int \dfrac{1 + \cos 2\theta}{2}\, d\theta = \dfrac{\theta}{2} + \dfrac{\sin 2\theta}{4} = \dfrac{\theta}{2} + \dfrac{\sin\theta\cos\theta}{2} = \dfrac{\arcsin x + x\sqrt{1 - x^2}}{2}$.

Thus, the area $= \dfrac{\arcsin x + x\sqrt{1 - x^2}}{2}\Big|_{-1}^{1} = \frac{1}{2}[(\arcsin 1 + 0) - (\arcsin(-1) + 0)]$
$= \frac{1}{2}[\pi - (-\pi)] = \frac{1}{2}(2\pi) = \pi$.