The Natural Logarithm Function and The Exponential Function

One specific logarithm function is singled out and one particular exponential function is singled out.

**Definition 1.** \(e = \lim_{{x \to 0}} (1 + x)^{1/x}\)

**Definition 2** (The Natural Logarithm Function). \(\ln x = \log_e x\)

**Definition 3** (The Exponential Function). \(\exp x = e^x\)

The following special cases of properties of logarithms and exponential functions are worth remembering separately for the natural log function and the exponential function.

- \(y = \ln x\) if and only if \(x = e^y\)
- \(\ln(e^x) = x\)
- \(e^{\ln x} = x\)

Suppose \(y = b^x\). By the properties of logarithms, we can write \(\ln y = \ln(b^x) = x \ln b\). It follows that \(e^{\ln y} = e^{x \ln b}\). But, since \(e^{\ln y} = y = b^x\), it follows that

- \(b^x = e^{x \ln b}\)

This important identity is very useful.

Similarly, suppose \(y = \log_b x\). Then, by the definition of a logarithm, it follows that \(b^y = x\). But then \(\ln(b^y) = \ln x\). Since \(\ln(b^y) = y \ln b\), it follows that \(y \ln b = \ln x\) and \(y = \frac{\ln x}{\ln b}\), yielding the following equally important identity.

- \(\log_b x = \frac{\ln x}{\ln b}\)

**Derivatives**

\[
\frac{d}{dx}(\ln x) = \frac{1}{x}
\]

\[
\frac{d}{dx}(e^x) = e^x
\]

If there are logs or exponentials with other bases, one may still use these formulas after rewriting the functions in terms of natural logs or the exponential function.

**Example:** Calculate \(\frac{d}{dx}(5^x)\)

**Solution:** Using the formula \(a^x = e^{x \ln a}\), write \(5^x\) as \(e^{x \ln 5}\). We can then apply the Chain Rule, writing:

- \(y = 5^x = e^{x \ln 5}\)
- \(y = e^u\)
\[ u = x \ln 5 \]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \ln 5 = 5^x \ln 5
\]

Example: Calculate \( \frac{d}{dx} (\log_7 x) \)

Solution: Using the formula \( \log_b x = \frac{\ln x}{\ln b} \), we write \( \log_7 x = \frac{\ln x}{\ln 7} \), so we can proceed as follows:

\[
y = \log_7 x = \frac{\ln x}{\ln 7} = \frac{1}{\ln 7} \cdot \ln x
\]

\[
y' = \frac{1}{\ln 7} \cdot \frac{d}{dx} (\ln x) = \frac{1}{\ln 7} \cdot \frac{1}{x} = \frac{1}{x \ln 7}
\]