Combinatorics

Definition 1 (Combinatorics). Combinatorics is the science of counting.

Theorem 1 (Fundamental Principle of Counting). If a sequence of choices are made and the first choice can be made in $n_1$ ways, the second in $n_2$ ways, the third in $n_3$ ways, and so on, then the entire sequence of choices can be made in $n_1 \cdot n_2 \cdot n_3 \ldots$ ways.

Definition 2 (Permutation). A permutation is a arrangement or list.

Definition 3 (Combination). A combination is a set.

Given a combination, there may be several permutations of the same elements.
Example: The set \( \{a, b, c\} \) yields the following permutations:

- a, b, c
- a, c, b
- b, a, c
- b, c, a
- c, a, b
- c, b, a

We may refer to permutations with or without replacement. In a permutation with replacement, the same element may appear more than once. In a permutation without replacement, no element may appear more than once.

**Theorem 2.** The number of permutations with replacement of length \( r \) of elements chosen from a set of size \( n \) is \( n^r \).

This theorem is almost obvious from the Fundamental Principle of Counting.
**Notation.** The number of permutations without replacement of length $r$ of elements chosen from a set of size $n$ is denoted by $P(n, r)$.

Alternate notations which may be found in other sources: $P_{n,r}$, $nP_r$.

**Theorem 3.** $P(n, r) = \frac{n!}{(n - r)!}$.

Here, $k!$, read $k$ factorial, means $k(k-1)(k-2)\cdots3\cdot2\cdot1$. In other words, $k!$ is the product of all the positive integers between 1 and $k$. $k$ must be an integer!
Notation (Combinations). The number of combinations of $r$ elements chosen from a set of size $n$ is denoted by $C(n, r)$.

Alternate notations which may be found in other sources: $C_{n,r}$, $_nC_r$, $\binom{n}{r}$.

We sometimes refer to $C(n, r)$ as $n$ choose $r$, since it may be thought of as the number of ways of choosing $r$ objects from a set of size $n$. 
Theorem 4. \( C(n, r) = \frac{n!}{r!(n-r)!} \).

Proof. Each combination of \( r \) elements gives rise to \( P(r, r) \) different permutations of the same elements. Thus, the number of permutations of size \( r \) is \( P(r, r) \) times the number of combinations of the same size.

It follows that \( P(n, r) = C(n, r)P(r, r) \). Since \( P(n, r) = \frac{n!}{(n-r)!} \) and \( P(r, r) = \frac{r!}{0!} = r! \), it follows that \( C(n, r) = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!} \). 

Example: The number of Gin Rummy hands is \( C(52, 10) = \frac{52!}{10!42!} \).