ON SELF-EMBEDDINGS OF COMPUTABLE LINEAR ORDERS

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Abstract. We show the existence of a computable linear order without nontrivial
0'-computable self-embedding.

1. Background and Theorems

A self-embedding of a linear ordering $\mathcal{L}$ is an order-preserving injection from $\mathcal{L}$
into itself. We call a self-embedding nontrivial if it is not the identity function.

One of the fundamental classical theorems on countable linear orderings is the

Dushnik-Miller Theorem on Countable Linear Orderings [DM40]. Let $\mathcal{L}$
be a countably infinite linear ordering. Then there is a nontrivial self-embedding
of $\mathcal{L}$.

One of the long-standing open questions on computable linear orders is the char-
acterization of those computable linear orders which admit a nontrivial computable
self-embedding in the following sense.

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Definition. Let $\mathcal{L} = \langle L, < \rangle$ be a linear ordering.

(i) An interval (or convex subset) of $\mathcal{L}$ is any subset $I \subseteq L$ such that $a, b \in I$ and $a < c < b$ implies $c \in I$. (So, in particular, an interval in our sense need not have endpoints.)

(ii) An interval $I$ is discrete if every $a \in I$ (except a possible left endpoint of $I$) has an immediate predecessor in $I$, and every $a \in I$ (except a possible right endpoint of $I$) has an immediate successor in $I$.

(iii) $\mathcal{L}$ is $\eta$-like if $L$ is infinite and no infinite interval of $\mathcal{L}$ is discrete, i.e., if $L$ is infinite but has no interval of order-type $\omega$ or $\omega^*$. (Here $\omega^*$ is $\omega$ under the reverse ordering, and $\eta$ is the order-type of the rational numbers.)

(iv) $\mathcal{L}$ is strongly $\eta$-like if $L$ is infinite and there is a fixed finite bound on the size of all discrete intervals of $\mathcal{L}$.

The open question is now contained in the following

Conjecture (folklore, see Downey [Do98, Conjecture 5.2]). Let $\mathcal{L}$ be an infinite computable linear ordering. Then $\mathcal{L}$ contains a strongly $\eta$-like interval iff for any computable copy $\mathcal{M} \cong \mathcal{L}$, there is a nontrivial computable self-embedding of $\mathcal{M}$.

We note right away that the left-to-right direction is trivial: Given any computable copy $\mathcal{M} = \langle M, < \rangle$ of a linear ordering with strongly $\eta$-like interval $I$, we can find a computable subset $J \subset I$ of order-type $\eta$; we can now effectively embed $\mathcal{M}$ into the subset $J$.

The other direction, however, has defied researchers for decades, and only partial results have been obtained thus far. These partial results seem to indicate that the solution to this question is very hard:

Theorem 1.

(i) (Downey [Do98, remark after Conjecture 5.2]) The above conjecture holds for computable $\eta$-like linear orderings (i.e., the equivalence holds when restricted to $\eta$-like linear orderings).

(ii) (Downey [Do98, Theorem 5.23]) The above conjecture does not hold uniformly: There is no effective procedure which, given (an index for) a computable linear ordering $\mathcal{L}$, produces (an index for) a computable copy $\mathcal{M} \cong \mathcal{L}$ such that if $\mathcal{L}$ contains no strongly $\eta$-like interval then there is no nontrivial computable self-embedding of $\mathcal{M}$.

(iii) (Downey [Do98, Theorem 5.24]) The above conjecture does not hold if we replace “isomorphic” by “$\Delta^0_2$-isomorphic”; in fact, there is an infinite computable linear ordering $\mathcal{L}$ without $\eta$-like interval such that for any computable copy $\mathcal{M} \cong_{\Delta^0_2} \mathcal{L}$, there is a nontrivial computable self-embedding of $\mathcal{M}$.

The following older result was the first theorem to show that the Dushnik-Miller Theorem on Countable Linear Orderings does indeed not hold effectively.

Theorem 2 (Hay/Rosenstein, see Rosenstein [Ro82]). There is a computable copy of $\omega$ without nontrivial computable self-embedding.
This result was extended as follows, which also allows the proof-theoretic characterization of the Dushnik-Miller Theorem on Countable Linear Orderings in the sense of reverse mathematics (see Simpson [Si99] for background on reverse mathematics).

**Theorem 3** (Downey/Lempp [DL99]). There is a computable copy $\mathcal{L}$ of $\omega$ such that any nontrivial self-embedding of $\mathcal{L}$ computes the halting problem $\emptyset'$. Thus, over the base theory RCA$_0$, the Dushnik-Miller Theorem on Countable Linear Orderings is equivalent to the theory ACA$_0$.

Since it is easy to see that any computable linear order has a nontrivial self-embedding computable in $\emptyset''$, the question naturally arises as to the exact upper bound on the possible complexity of a nontrivial self-embedding. The main result of the present paper partially answers this question as follows:

**Theorem 4.** There is a computable linear ordering $\mathcal{L}$ such that no nontrivial self-embedding of $\mathcal{L}$ is computable from the halting problem $\emptyset'$.

It remains open whether Theorem 3 can be improved to coding $\emptyset''$ (in place of $\emptyset'$) into any nontrivial self-embedding of a computable linear order. At the same time, Theorem 4 is further evidence that the solution to the above conjecture is very difficult.

2. The Proof of Theorem 4

We build a computable linear order $\mathcal{L} = \langle \omega, <_\mathcal{L} \rangle$ and ensure that no function computable in $\emptyset'$ is a nontrivial self-embedding of $\mathcal{L}$. (Our notation generally follows Soare [So87].)

For this, we first fix an effective listing $\{\Psi^\emptyset_e\}_{e \in \omega}$ of all functions partial computable in $\emptyset'$ and an effective listing of partial computable functions $\{\psi_e\}_{e \in \omega}$ such that for all $e \in \omega$, $\Psi^\emptyset_e = \lim_u \psi_{e,u}$. Without loss of generality, we may assume that each $\psi_{e,u}$ is a total function (although the limit in $u$ may diverge for some arguments).

We need to meet the following

**Requirement:**

$$\mathcal{R}_e : \Psi^\emptyset_e \text{ is not a nontrivial self-embedding of } \mathcal{L}.$$  

Rather than meeting this requirement all at once, we will try to meet, for all $e, x \in \omega$, the following subrequirement

$$\mathcal{R}_{e,x} : \Psi^\emptyset_e \text{ is not a self-embedding of } \mathcal{L} \text{ moving } x.$$  

**Remark.** Note that, for any $e \in \omega$, we actually need to meet the subrequirement $\mathcal{R}_{e,x}$ only for at least one $x$ in each maximal discrete interval of $\mathcal{L}$. (This will be crucial to our construction later.)
Strategy for $\mathcal{R}_{e,x}$. For simplicity, abbreviate $\psi_{e,u}(\psi_{e,u}(x))$ as $\psi_{e,u}^2(x)$. As the construction proceeds, we consider $\psi_{e,u}(x)$ and $\psi_{e,u}^2(x)$ for larger and larger $u$ and proceed as follows:

1. Whenever $\psi_{e,u}(x) \neq \psi_{e,u-1}(x)$ or $\psi_{e,u}^2(x) \neq \psi_{e,u-1}^2(x)$, we restart, canceling any restraint (since $\Psi_{e}^\alpha(x)$ or $\Psi_{e}^\alpha'(x)$ appear undefined).
2. Unless $x < \mathcal{L} \psi_{e,u}(x) < \mathcal{L} \psi_{e,u}^2(x)$ or $x > \mathcal{L} \psi_{e,u}(x) > \mathcal{L} \psi_{e,u}^2(x)$, we do nothing (since $\Psi_{e}^\alpha$ does not appear to move $x$, or does not appear to preserve $<\mathcal{L}$).
3. By symmetry, we now assume that $x < \mathcal{L} \psi_{e,u}(x) < \mathcal{L} \psi_{e,u}^2(x)$. We then ensure that $|[x, \psi_{e,u}(x)]| > |[\psi_{e,u}(x), \psi_{e,u}^2(x)]|$ by restraining the interval $[\psi_{e,u}(x), \psi_{e,u}^2(x)]$ (which is finite at this stage, of course) i.e., not allowing any new points into this interval; and by adding enough points into the interval $[x, \psi_{e,u}(x)]$.

Outcomes of the $\mathcal{R}_{e,x}$-strategy. An individual $\mathcal{R}_{e,x}$-strategy may have the following outcomes (keeping in mind that we assume that each $\psi_{e,u}$ is total):

- $\infty$ The $\mathcal{R}_{e,x}$-strategy restarts infinitely often in Step 1: Then at least one of $\Psi_{e}^\alpha(x)$ and $\Psi_{e}^\alpha'(x)$ is not defined, the requirement $\mathcal{R}_e$ is met, and the $\mathcal{R}_{e,x}$-strategy has no permanent effect on our construction.
- id The $\mathcal{R}_{e,x}$-strategy is eventually stuck waiting in Step 2 forever: Then $\Psi_{e}^\alpha$ does not move $x$ or does not preserve $<\mathcal{L}$, the subrequirement $\mathcal{R}_{e,x}$ is met, and the $\mathcal{R}_{e,x}$-strategy has no permanent effect on our construction.
- fin The $\mathcal{R}_{e,x}$-strategy is eventually stuck waiting in Step 3 forever: Then $\Psi_{e}^\alpha$ cannot preserve $<\mathcal{L}$ since $|[x, \Psi_{e}^\alpha(x)]| > |[\Psi_{e}^\alpha'(x), \Psi_{e}^\alpha(\Psi_{e}^\alpha'(x))]|$, the requirement $\mathcal{R}_e$ is met, and the $\mathcal{R}_{e,x}$-strategy has only finite effect on our construction.

Thus under outcomes $\infty$ and $\text{fin}$, the overall requirement $\mathcal{R}_e$ is met, while under outcome id, only the subrequirement $\mathcal{R}_{e,x}$ is met. We agree that the apparent outcome of the $\mathcal{R}_{e,x}$-strategy at a given stage is $\infty$ if the strategy restarts at that stage under Step 1; and id or fin if the strategy is waiting at Step 2 or 3, respectively, at that stage.

Outcomes of the $\mathcal{R}_e$-requirement. For each $e \in \omega$ and along any path $p$ of the tree of strategies $T$ (to be defined precisely later), either (i) we have an $\mathcal{R}_{e,x}$-strategy $\alpha_x$ for at least one $x$ in each maximal discrete interval of $\mathcal{L}$ such that each $\alpha_x$ has outcome id along the path $p$; or (ii) there is a longest $\mathcal{R}_{e,x}$-strategy $\alpha$ for some $x$ such that $\alpha$ has outcome $\infty$ or $\text{fin}$ along the path $p$. In the former case, if $\Psi_{e}^\alpha$ is indeed a self-embedding of $\mathcal{L}$, then it is the identity; in the latter case, $\Psi_{e}^\alpha$ is partial or does not preserve $<\mathcal{L}$.

Conflicts between strategies. The only possible conflict is between an $\mathcal{R}_{e',x}$-strategy $\alpha' \in T$, wanting to restrain an interval, and an $\mathcal{R}_{e,x}$-strategy $\alpha > \alpha'$, wanting to insert numbers into that interval. Note that for a fixed $\alpha$ along the true path, there can be at most finitely many such $\alpha'$ wanting to restrain an interval $[a_{\alpha'}, b_{\alpha'}]$, say. This means that $x$ is in one (or several) of these intervals $[a_{\alpha'}, b_{\alpha'}]$, and that as long as $\alpha$ is not initialized, these intervals do not increase in size. Let $[a, b]$ be the smallest interval containing $x$ and either containing, or disjoint from,
all intervals \([a_\alpha', b_\alpha']\). (This interval is the restraint on points near \(x\) that \(\alpha\) has to deal with.)

We now recall our remark at the beginning of this section: We only need to meet subrequirement \(R_{e,x}\) for at least one \(x\) in each maximal discrete interval of \(L\). So, to meet subrequirement \(R_{e,x}\), we essentially “replace” \(x\) by \(a\) and \(b\) and proceed as follows:

1. Whenever \(\psi_{e,u}(a) \neq \psi_{e,u-1}(a)\), \(\psi_{e,u}^2(a) \neq \psi_{e,u-1}^2(a)\), \(\psi_{e,u}(b) \neq \psi_{e,u-1}(b)\), or \(\psi_{e,u}^2(b) \neq \psi_{e,u-1}^2(b)\), we restart, canceling any restraint (since \(\Psi^\theta_\psi(a), \Psi^\theta_\psi^\prime(\Psi^\theta_\psi(a))\), \(\Psi^\theta_\psi^\prime(b)\) or \(\Psi^\theta_\psi^\prime(\Psi^\theta_\psi^\prime(b))\) appear to be undefined).

2. Unless \(b < L \psi_{e,u}(b) < L \psi_{e,u}^2(b)\) or \(a > L \psi_{e,u}(a) > L \psi_{e,u}^2(a)\), we do nothing (since \(\Psi^\theta_\psi^\prime\) does not appear to move \(b\) to the right, or \(a\) to the left, or does not appear to preserve \(< L\)). (Note here that if the restraint on \([a, b]\) is permanent then if \(\Psi^\theta_\psi^\prime\) moves \(x\) to the right, it must also move \(b\) to the right, and if it moves \(x\) to the left, it must also move \(a\) to the left.)

3. By symmetry, we now assume that \(b < L \psi_{e,u}(b) < L \psi_{e,u}^2(b)\). We then ensure that \([b, \psi_{e,u}(b)] > [\psi_{e,u}(b), \psi_{e,u}^2(b)]\) by restraining the interval \([\psi_{e,u}(b), \psi_{e,u}^2(b)]\) (which is finite at this stage, of course), and adding enough points into the interval \([b, \psi_{e,u}(b)]\). (Note here that by our assumption on \(b\), there is no restraint on points immediately to the right of \(b\), so there is no conflict with higher-priority strategies.)

Construction. It is now easy to describe the full construction. We first define the set of outcomes \(\Lambda = \{\infty < \Lambda \text{id} < \Lambda \text{fin}\}\) and the tree of strategies \(T = \Lambda^{< \omega}\) (with the lexicographical order \(<\) induced by the order \(<\)). We then assign subrequirements to nodes of \(T\) by recursion as follows: Fix any effective ordering of all subrequirements \(R_{e,x}\) of order-type \(\omega\) and assign the highest-priority subrequirement to the root of \(T\). Call a subrequirement \(R_{e,x}\) satisfied along \(\alpha \in T\) if there is some \(R_{e,x}\)-strategy \(\alpha' \subset \alpha\), or if, for some \(x' \neq x\), there is an \(R_{e,x'}\)-strategy \(\alpha'\) with \(\alpha' < (\infty) \subseteq \alpha\) or \(\alpha' < \text{fin} \subseteq \alpha\). (The intuition here is that then either \(R_{e,x'}\) or \(R_e\) is satisfied along \(\alpha\) by this \(\alpha'\).) We then assign to \(\alpha\) the highest-priority subrequirement not satisfied along \(\alpha\).

A strategy \(\alpha \in T\) is initialized by setting its parameter \(u\) to 0 and canceling its restraint. At stage 0, all strategies are initialized.

Each stage \(s > 0\) consists of substages \(t \leq s\), at which a strategy \(\alpha \in T\) of length \(t\) is eligible to act. At substage \(t\), the \(R_{e,x}\)-strategy \(\alpha\) eligible to act then proceeds as described above (increasing its parameter \(u\) by 1 at each stage at which \(\alpha\) is eligible to act), taking into account the restraint of all the finitely many strategies \(\alpha' < \alpha\) currently restraining intervals. (If \(x\) has not yet appeared in \(L\) by this stage, we simply end the stage.) At the end of stage \(s\), we add an additional point to the right of all current points in \(L\) (to ensure that \(L\) is infinite) and initialize all strategies > the last strategy eligible to act at stage \(s\).

Verification. By the action at the end of each stage, the linear ordering \(L\) is infinite (and we may assume that its domain is all of \(\omega\)). Thus, for each \(t\), stage \(s\) is eventually not ended before substage \(t\) due to lack of points. So we can define the true path of the construction \(f \in [T]\) as the leftmost infinite path through \(T\) along which all strategies are eligible to act infinitely often.
For each \( e \in \omega \), there are now three possibilities:

1. There is a longest \( R_{e,x} \)-strategy \( \alpha \) (for some \( x \in \omega \)) and \( \alpha^\sim(\infty) \subset f \): Then, by definition of initialization and of the true path, \( \alpha \) is not initialized after some stage \( s_\alpha \), say. So, after stage \( s_\alpha \), \( \alpha \) meets the requirement \( R_e \) as in the basic module above by showing that \( \Psi_\alpha \) is partial.

2. There is a longest \( R_{e,x} \)-strategy \( \alpha \) (for some \( x \in \omega \)) and \( \alpha^\sim(\text{fin}) \subset f \): Then, by definition of initialization and of the true path, \( \alpha \) is not initialized after some stage \( s_\alpha \), say. So, after stage \( s_\alpha \), \( \alpha \) meets the requirement \( R_e \) as in the basic module above by showing that \( \Psi_\alpha \) does not preserve \( <_\mathcal{L} \), handling the conflict with the finitely many higher-priority strategies \( \alpha' < \alpha \) as explained above.

3. Otherwise: Then, by definition of the tree of strategies, for each \( x \), there is an \( R_{e,x} \)-strategy \( \alpha_x \subset f \), and for each \( x \), \( \alpha_x^\sim(\text{id}) \subset f \). By definition of initialization and of the true path, \( \alpha_x \) is not initialized after some stage \( s_x \), say. Thus, for each \( x \), after stage \( s_x \), \( \alpha_x \) shows that \( \Psi_\alpha \) does not preserve \( <_\mathcal{L} \) or that \( x = \Psi_\alpha(x) \), which implies that \( \Psi_\alpha \) is not a nontrivial computable self-embedding of \( \mathcal{L} \).

This concludes the proof of Theorem 4.

References


