

FRIEDBERG NUMBERINGS OF FAMILIES OF n -COMPUTABLY ENUMERABLE SETS

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ABSTRACT. We establish a number of results on numberings, in particular on Friedberg numberings, of families of d. c. e. sets:

(1) There exists a Friedberg numbering of the family of all d. c. e. sets. We also show that this result, patterned on Friedberg's famous theorem for the family of all c. e. sets, holds for the family of all n -c. e. sets for any $n > 2$.

(2) There exists an infinite family of d. c. e. sets without a Friedberg numbering.

(3) There exists an infinite family of c. e. sets with a numbering (as a family of d. c. e. sets) which is unique up to equivalence.

(4) There exists a family of d. c. e. sets with a least numbering (under reducibility) such that this numbering is a Friedberg numbering but not the only numbering (modulo reducibility).

1. THE THEOREMS

In one of the early fundamental papers of classical computability theory, Friedberg [Fr58] constructed an effective enumeration of the family of all computably enumerable sets of nonnegative integers without repetition, i. e., he built a uniformly computably enumerable sequence of sets $\{\alpha_n\}_{n \in \omega}$ such that each computably enumerable set occurs exactly once in this sequence.

This theorem can be viewed as an example of a result in the theory of numberings, a field initiated by Kolmogorov in the mid-1950's, which has since then been pursued

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mainly in the (former) Soviet Union, in particular by the Novosibirsk school under Mal'tsev and Ershov. (See Ershov [Er73-77, Er77, Er99] for more background. There is also some work in the 1960's by Lachlan [La65-67], Pour-El and coauthors [PH64, PP65], and others.)

A numbering tries to enable the algorithmic study of a (countable) family S of objects by giving "names" (i. e., integer indices) to the objects in S . More precisely, a *numbering* of S is a map ν from the set ω of natural numbers *onto* the family S . Of course, an object in S can have many "names" under ν , i. e., ν is generally not assumed to be 1-1. If, however, ν is 1-1, it is usually called a *Friedberg numbering* due to the above-mentioned result of Friedberg, which was the first example of a Friedberg numbering. Friedberg numberings play an important role in the theory of numberings since they are minimal under reducibility on the collection of numberings of a family S . (This *reducibility*, for numberings ν, μ of S , is defined by $\nu \leq \mu$ iff there is a computable function f such that $\nu = \mu \circ f$, i. e., from any ν -index of an object in S one can effectively compute a μ -index of this object. One can then define an *equivalence* relation on the collection of numberings of S by setting $\nu \equiv \mu$ iff $\nu \leq \mu$ and $\mu \leq \nu$. These equivalence classes now form a natural upper semilattice under the ordering induced by \leq ; and Friedberg numberings are only found in minimal elements of this semilattice.)

A natural extension of the notion of a computably enumerable set was defined by Putnam [Pu65]: Call a set $A \subseteq \omega$ 1-computably enumerable if it is computably enumerable; and $(n + 1)$ -computably enumerable if it is of the form $A_0 - A_1$ where A is computably enumerable and A_1 is n -computably enumerable. Equivalently, a set $A \subseteq \omega$ is n -computably enumerable if there is a uniformly computable sequence of sets $\{A_s\}_{s \in \omega}$ such that for all x ,

$$\begin{aligned} x &\notin A_0, \\ A(x) &= \lim_s A_s(x), \text{ and} \\ |\{s \in \omega \mid A_s(x) \neq A_{s+1}(x)\}| &\leq n. \end{aligned}$$

Given a family S of n -computably enumerable sets, we call a numbering ν of S *computable* if the relation " $x \in \nu(e)$ " is n -computably enumerable, i. e., if the sequence $\{\nu(e)\}_{e \in \omega}$ is uniformly n -computably enumerable. (Note that this notion depends not only on the family S but also on n since S might consist of $(n - 1)$ -computably enumerable sets only.)

In this notation, Friedberg's above-mentioned result now states that there is a computable Friedberg numbering of the family of all computably enumerable sets. Surprisingly, the question of whether, for any fixed $n > 1$, there is a Friedberg numbering of the family of all n -computably enumerable sets has thus far been open. We answer this question in the affirmative by the following

Theorem 1. *For any $n > 1$, there is an effective enumeration of the family of all n -computably enumerable sets without repetition. In other words, there is a computable Friedberg numbering of the family of all n -computably enumerable sets.*

Lachlan [La65] and Pour-El and Putnam [PP65] gave an example of an infinite family of c. e. sets without Friedberg numbering: For any noncomputable c. e. set

A , the family $\{\{2n, 2n + 1\} \mid n \in A\} \cup \{\{2n\}, \{2n + 1\} \mid n \notin A\}$ is computable but has no computable Friedberg numbering. Goncharov [Go80] gave examples of computable families with any fixed finite number of nonequivalent Friedberg numberings. Mal'cev [Mf65] and Kummer [K90] gave sufficient conditions on families with Friedberg numberings. In the papers [Go82, GYY93], many results were established about families with infinitely many nonequivalent Friedberg numberings. Goncharov [Go83] proved that a family with a computable Friedberg numbering which is not the least numbering has infinitely many positive computable numberings.

Theorem 2.

- (1) *There exists an infinite family of d. c. e. sets without Friedberg numbering.*
- (2) *There exists an infinite family of c. e. sets which (considered as a family of d. c. e. sets) has a unique numbering (up to equivalence).*
- (3) *There exists a family of d. c. e. sets with a least numbering (under reducibility) such that this numbering is a Friedberg numbering but not the only numbering (modulo reducibility).*

The rest of the paper is devoted to the proof of our theorems.

2. THE PROOF OF THEOREM 1

Our proof is loosely modeled on Friedberg's proof, as presented in Odifreddi [Od89]. (Note, however, an error in the proof in [Od89]: In the notation there, a least index e of a computably enumerable set W need not have a follower if W is finite of the form $[0, x]$ for some x .) We first present the proof of Friedberg's theorem (i. e., the case $n = 1$) in a way that can be easily generalized to arbitrary $n \geq 1$.

Proof for $n = 1$. Suppose we are given a computable numbering $\{\alpha_n\}_{n \in \omega}$ of the family S of all computably enumerable sets. Without loss of generality, we assume that $\alpha_0 = \omega$. We now build a computable numbering $\{\beta_n\}_{n \in \omega}$ of S and a \emptyset' -partial computable function f (approximated by uniformly partial computable functions f_s in the sense that $f(n) \downarrow = m$ if $f_s(n) = m$ for cofinitely many s , and $f(n)$ is undefined otherwise). We meet the following

Requirements:

- (i) If $\alpha_n = \alpha_{n'}$ for some $n' < n$ then $f(n)$ is undefined.
- (ii) If $\alpha_n \neq \alpha_{n'}$ for all $n' < n$ then either $f(n)$ is defined and $\alpha_n = \beta_{f(n)}$; or α_n is of the form $[0, x]$ for some x , and there is $m \in \omega - \text{ran}(f)$ such that $\alpha_n = \beta_m$.
- (iii) Any set β_m with $m \notin \text{ran}(f)$ is of the form $[0, x]$ for some x .
- (iv) For any set of the form $[0, x]$ for some x , there is a unique m with $\beta_m = [0, x]$.

Construction: At stage $s = 0$, we define $\beta_0 = \omega$ and $f(0) = f_0(0) = 0$, while $f_0(n)$ is undefined for all $n > 0$.

At a stage $s + 1$, we perform the following steps:

Step 1: If $f_s(n)$ is defined and for some $n' < n$,

$$\alpha_{n',s} \upharpoonright (f_s(n) + 1) = \alpha_{n,s} \upharpoonright (f_s(n) + 1)$$

(i. e., if n does not appear to be the least index for α_n , then let $f_{s+1}(n)$ be undefined (and keep $f_s(n)$ permanently out of the range of f from now on).

Step 2: If $f_s(n)$ is defined, $n > 0$, and, for some $s' < s$ and some $m \in \text{ran}(f_{s'}) - \text{ran}(f_s)$,

$$\beta_{m,s} \upharpoonright (f_s(n) + 1) = \beta_{f_s(n),s} \upharpoonright (f_s(n) + 1)$$

(i. e., if the set β_m seems to appear twice in the β -sequence of sets, including once as a set with index no longer in the range of f), then let $f_{s+1}(n)$ be undefined (and keep $f_s(n)$ permanently out of the range of f from now on).

Step 3: If $f_s(n)$ is defined but $f_{s+1}(n)$ is undefined (i. e., if $f(n)$ just became undefined via Step 1 or Step 2), then for each such n (in increasing order of n), set

$$\beta_{f_s(n)} = \beta_{f_s(n),s+1} = [0, x]$$

for some x larger than any number mentioned thus far in the construction.

Step 4: If $f_s(n)$ is undefined for $n \leq s$, then for each such n (in increasing order of n), let $f_{s+1}(n)$ be the least m not in $\bigcup_{s' \leq s} \text{ran}(f_{s'})$ and not equal to $f_{s+1}(n')$ for some $n' < n$.

Step 5: If $f_{s+1}(n)$ is defined then let $\beta_{f_{s+1}(n),s+1} = \alpha_{n,s+1}$.

Verification: We first note that since for each m there is at most one n such that $f_s(n) = m$ at some stage s , Step 5 can be carried out since no number has to be removed from $\beta_{f_{s+1}(n)}$ to carry out Step 5. Similarly, since x is chosen large in Step 3, this step can be carried out without removing numbers from $\beta_{f_s(n)}$.

We now verify the satisfaction of the above requirements:

(i) If $\alpha_n = \alpha_{n'}$ for some $n' < n$ then $f_s(n)$ is undefined for infinitely many s by Step 1.

(ii) If $\alpha_n \neq \alpha_{n'}$ for all $n' < n$ then $f(n)$ becomes undefined via Step 1 at most finitely often. If $f(n)$ becomes undefined via Step 2 for the same m infinitely often then $\alpha_n = \beta_m$ as desired. Otherwise, since α_n is computably enumerable, $\alpha_{n,s} = [0, x]$ at various stages s for larger and larger x ; thus $\alpha_n = \omega$, and so $n = 0$ and Step 2 never applies to n .

(iii) This is immediate by Step 4.

(iv) Fix x . Steps 2 and 4 ensure that there is at most one m such that $\beta_m = [0, x]$. Fix n least such that $\alpha_n = [0, x]$. Then either $f(n)$ is defined and $\beta_{f(n)} = [0, x]$; or else we can argue as in (ii) above that there is some m such that $\beta_m = [0, x]$.

Proof for $n > 1$. We merely note some minor modifications to the above needed for $n > 1$: Fix a computable numbering $\{\alpha_n\}_{n \in \omega}$ of the family S of all n -computably enumerable sets. Without loss of generality, we assume that $\alpha_0 = \omega$ if n is odd, and that $\alpha_0 = \emptyset$ if n is even. We again build a computable numbering $\{\beta_n\}_{n \in \omega}$ of S and a \emptyset' -partial computable function f , meeting the same requirements (i)–(iv) as above except that in (iii) and (iv), we replace $[0, x]$ by $\omega - [0, x]$ in the case that n is even.

Construction: We perform Steps 1–5 as above, except that in Step 3, we replace $[0, x]$ by $\omega - [0, x]$ in the case that n is even.

Verification: We proceed as above, but note that we need a new argument that Step 3 can be carried out as prescribed. But this holds since x is larger than any

number mentioned thus far in the construction. So, in the case that n is even, Step 3 does not add numbers into $\beta_{f_s(n)}$; and in the case that n is odd, Step 3 does not remove numbers from $\beta_{f_s(n)}$.

Concluding Remarks. We remark in closing that, as for the case $n = 1$, the above construction can be adapted, for any fixed $n > 1$, to any uniformly computable family S of n -computably enumerable sets as long as S contains all finite sets (if n is odd) or all cofinite sets (if n is even, respectively). If ω (if n is odd) or the empty set (if n is even, respectively) is not in S , we can add it to S and then later change the numbering β by removing it again, which is possible since ω , or the empty set, respectively, appears only once in $\{\beta_m\}_{m \in \omega}$.

3. THE PROOFS FOR THEOREM 2

The proofs for Theorem 2 are fairly simple constructions.

The proof of part (1). We fix an effective list of all computable numberings $\{\mu_e\}_{e \in \omega}$ of d. c. e. sets and build a computable numbering ν of d. c. e. sets (enumerating a family S of d. c. e. sets).

For each $e \in \omega$, we act as follows:

1. Enumerate $2e$ into $\nu(2e)$, and $2e + 1$ into $\nu(2e + 1)$.
2. Wait for a stage s and distinct indices i and j such that $2e \in \mu_{e,s}(i)$ and $2e + 1 \in \mu_{e,s}(j)$.
3. Extract $2e$ and $2e + 1$ from $\nu(2e)$ and $\nu(2e + 1)$, respectively.
4. Wait for $2e$ and $2e + 1$ to leave $\mu_e(i)$ and $\mu_e(j)$, respectively, by a stage $s' > s$, say.
5. Enumerate both $2e$ and $2e + 1$ into $\nu(e')$ for all $e' \neq 2e, 2e + 1$.

Now suppose that μ_e is a numbering of a family T of d. c. e. sets. If a stage s and indices i and j as above do not exist for μ_e then S contains two distinct sets containing $2e$ and $2e + 1$, respectively, but T does not; so $S \neq T$. If stage s exists as above but stage s' does not then T contains a set containing either $2e$ or $2e + 1$ but S does not; so again $S \neq T$. Finally, if stage s' exists as above then the only set in S not a superset of $\{2e, 2e + 1\}$ is $\nu(2e) = \nu(2e + 1)$, but $\{2e, 2e + 1\} \not\subseteq \mu_e(i), \mu_e(j)$, so $S = T$ implies $\mu_e(i) = \mu_e(j)$ for distinct indices i and j ; thus μ_e cannot be a Friedberg numbering.

The proof of part (2). We again fix an effective list of all computable numberings $\{\mu_e\}_{e \in \omega}$ of d. c. e. sets and build a computable numbering ν of d. c. e. sets (enumerating a family S of c. e. sets).

For each $e \in \omega$, we act as follows:

1. Enumerate the coded pair $\langle n, e \rangle$ into $\nu(n)$ for each n .
2. For all $k, n \in \omega$ for which $f_e(k)$ and $g_e(n)$ are currently undefined, if $\langle n, e \rangle$ enters $\mu_e(k)$ then define $f_e(k) = n$ and $g_e(n) = k$ (indicating our prediction that $\mu_e(k) = \nu(n)$).
3. If ever $\langle n, e \rangle$ leaves $\mu_e(k)$ while $f_e(k) = n$ and $g_e(n) = k$, then enumerate $\langle n, e \rangle$ into $\nu(n')$ for all $n' \neq n$ (so that μ_e cannot be a computable numbering of S as a family of d. c. e. sets).

Now suppose that μ_e is a numbering of S as a family d. c. e. sets. First suppose that Step 3 above never applies. Then each $\nu(n)$ contains exactly one element of the form $\langle n', e \rangle$, namely, $\langle n, e \rangle$. Thus, if μ_e is a numbering of S , then each set $\mu_e(k)$ must contain exactly one element of the form $\langle n, e \rangle$, and for each n there must be at least one k such that $\mu_e(k)$ contains $\langle n, e \rangle$; thus f_e and g_e are computable reductions witnessing that ν and μ_e are equivalent numberings. On the other hand, if Step 3 ever applies to $\langle n, e \rangle$ and k , say, then $\mu_e(k)$ does not contain $\langle n, e \rangle$ but $\nu(n')$ does for all n' , so μ_e cannot be a computable numbering of S as a family of d. c. e. sets.

The proof of part (3). We fix effective lists of all computable numberings $\{\mu_e\}_{e \in \omega}$ of d. c. e. sets and of all partial computable functions $\{h_i\}_{i \in \omega}$. We build two computable numberings σ and τ of d. c. e. sets (enumerating the same family S of d. c. e. sets) as well as a computable function f and a \emptyset' -computable function g (approximated by a uniformly computable sequence of functions $\{g_s\}_{s \in \omega}$), meeting the following requirements:

$$(3.1) \quad \sigma = \tau \circ f,$$

$$(3.2) \quad \tau = \sigma \circ g,$$

$$(3.3) \quad \forall i (\tau \neq \sigma \circ h_i), \text{ and}$$

$$(3.4) \quad \forall e (\text{ran } \mu_e = \text{ran } \sigma \rightarrow \exists \text{ computable function } k_e (\sigma = \mu_e \circ k_e)).$$

Note that requirements (3.1) and (3.2) ensure that σ and τ enumerate the same family of d. c. e. sets S and that $\sigma \leq \tau$. Requirement (3.3) now implies that $\sigma < \tau$, while requirement (3.4) ensures that σ represents the least of the Rogers semilattice of S . Our construction will also ensure that σ is a Friedberg numbering.

Requirement (3.3) is met by diagonalization: We fix an index $j \notin \text{ran } f$ and wait for $h_i(j)$ to be defined. Then we enumerate $\langle 0, 2i \rangle$ into $\tau(j)$ and keep $\langle 0, 2i \rangle$ out of $\sigma(h_i(j))$.

Requirement (3.4) is met by strongly using the fact that the sets in S are d. c. e. as follows: We enumerate $\langle n, 2e + 1 \rangle$ into $\sigma(n)$ for each n . Now, for each n , we wait for $\langle n, 2e + 1 \rangle$ to appear in $\mu_e(j_n)$ for some j_n (distinct from all $j_{n'}$ found previously). We now extract $\langle n, 2e + 1 \rangle$ from $\sigma(n)$ (so that no σ -set contains $\langle n, 2e + 1 \rangle$ at this point). When $\langle n, 2e + 1 \rangle$ leaves $\mu_e(j_n)$, then we set $k_e(n) = j_n$ and enumerate $\langle n, 2e + 1 \rangle$ into $\sigma(n')$ for all $n' \neq n$. (Note that if μ_e is indeed a numbering of S then for each n , j_n must eventually be defined and later $\langle n, 2e + 1 \rangle$ must leave $\mu_e(j_n)$. But then $\langle n, 2e + 1 \rangle \notin \mu_e(j_n)$, so $\sigma(n) = \mu_e(j_n)$ since $\sigma(n)$ is now the only σ -set not containing $\langle n, 2e + 1 \rangle$.)

Requirements (3.1) and (3.2) are met by directly constructing f and g : To build f , we simply match up each σ -set with a τ -set, leaving an infinite computable set J of τ -indices outside the range of f (so that we can use these $j \in J$ to meet requirement (3.3)). Similarly, to build g , we match each τ -set with a σ -set. We now copy $\sigma(i)$ into $\tau(f(i))$ and vice versa. We also copy $\tau(j)$ into $\sigma(g(j))$ unless requirement (3.3) prohibits this since we need to enumerate into $\tau(j)$ but keep it out of $\sigma(g(j))$ (i. e., $\sigma(g(j))$, for the current value of $g(j)$, and $\sigma(h_i(j))$ are supposed

to be the same set). In that case, we simply change $g(j)$ to a new σ -index i' never used before so that $\sigma(i')$ can copy $\tau(j)$.

It is now not hard to see how to combine these strategies into a finite-injury priority argument, the details of which we leave to the reader.

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