**Math 2142 Homework 5: Due Wednesday February 25**

**Problem 1.** Solve the following separable initial value problems.

\[
\frac{dy}{dx} = (1 - 2x)y^2 \quad \text{with} \quad y(0) = -1/6
\]

\[
\frac{dy}{dx} = xy^3(1 + x^2)^{-1/2} \quad \text{with} \quad y(0) = 1
\]

**Problem 2.** Find the general solutions for the following separable differential equations. For the second one, you do not need to solve explicitly for \(y\).

\[
\frac{dy}{dx} = \frac{x^2}{y(1 + x^3)} \quad \text{(you can assume that } x > 0)\]

\[
\frac{dy}{dx} = \frac{2x}{ye^y - x^2 ye^y} \quad \text{(you can assume that } -1 < x < 1)\]

For the next problem, you will need to solve equations of the form \(y^2 + by + c = f(x)\) explicitly for \(y\). Since the left side of this equation is a quadratic function of \(y\), you can solve for \(y\) by completing the square on the left side to get something of the form \((y+b/2)^2 + d = f(x)\). From here, you can directly solve for \(y\).

**Problem 3(a).** Solve the following algebraic equation explicitly for \(y\).

\[y^2 - 2y + 4 = x^4 + x^2 + 6\]

**3(b).** Solve the following separable initial value problem and give the solution explicitly as a function \(y(x)\).

\[
\frac{dy}{dx} = \frac{3x^2 + e^x}{2y - 4} \quad \text{with} \quad y(0) = 1
\]

**Problem 4.** A tank contains 1000 liters of water with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 liters per minutes and the mixture is drained from the bottom at the same rate. Assume that the solution is kept thoroughly mixed, how many kilograms of salt are in the tank after \(t\) minutes?

**Problem 5.** Newton’s Law of Cooling states that the rate of cooling (or heating) of an object is proportional to the temperature difference between the object and its surroundings. In other words, if \(T(t)\) is the temperature of the object at time \(t\) and \(T_s\) is the constant temperature of its surroundings, then Newton’s Law of Cooling says that \(T(t)\) satisfies

\[
\frac{dT}{dt} = k(T(t) - T_s)
\]
for some constant \( k \).

5(a). A can of soda at 72 degrees is placed in a refrigerator where the temperature is 44 degrees. Use Newton’s Law of Cooling to set-up and solve a differential equation for \( T(t) \), the temperature of soda after \( t \) minutes. (The constant \( k \) will appear in your answer but you can solve for the constant of integration using the initial temperature of the can.)

5(b). After 20 minutes, the temperature of the can is 61 degrees. Use this information to find the constant \( k \).

5(c). How long does it take for the soda to cool to 50 degrees?

Problem 6. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction of people who have heard the rumor and the fraction of the people who have not heard the rumor. If \( y(t) \) denotes the fraction of people who have heard the rumor after \( t \) days, then the modeling differential equation is

\[
\frac{dy}{dt} = ky(1 - y)
\]

where \( k \) is the proportionality constant.

6(a). Find the general solution to this separable differential equation. (You can assume that \( 0 < y < 1 \), and hence \( 0 < 1 - y < 1 \) as well. The constant \( k \) will appear in your solution in addition to the constant of integration.)

6(b). Suppose at \( t = 0 \), one person is a class of 25 has heard a rumor and at \( t = 1 \), two people in the class have heard the rumor. Use this information to find the value of the constant of integration and the constant \( k \).

6(c). When will 90% of the class have heard the rumor?

Problem 7. Find the general solution for the following linear differential equations. If an initial condition is given, find the specific solution as well.

\[
\frac{dy}{dx} + 2xy = 4e^{-x^2} \quad \text{with} \quad y(0) = 3
\]

\[
\frac{dy}{dt} = \frac{y}{t} + 2t^2 \quad \text{(you can assume } t \text{ is positive)}
\]

\[
\frac{dy}{dx} + y = 3x
\]

\[
\frac{dy}{dx} + 3y = \sin 2x
\]

Problem 8. A tank contains 100 liters of pure water. Brine containing 0.4 kilograms of salt per liter is added to the tank at a rate of 5 liters per minute. The solution in the tank is kept thoroughly mixed and is drained at a rate of 3 liters per minute. How many kilograms of salt are in the tank after \( t \) minutes?
Practice Problems

Exercises 8.5: 1, 2, 4, 5.

Exercises 8.24: 1-9