Math 2142 Homework 4: Due Friday February 20

Problem 1. Do the following computational exercises from Chapter 9.

1(a). Exercises 9.6, 1(a)-(h).
1(b). Exercises 9.6, 2(a)(c)(f)
1(c). Exercises 9.6, 3(a)(c)(e)(f)
1(d). Exercises 9.10, 1(b)(e)(f)

Problem 2(a). Prove that for all $z_1, z_2 \in \mathbb{C}$, we have $|z_1 z_2| = |z_1||z_2|$ and $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$.

2(b). Use 2(a) and induction to prove that for all $z \in \mathbb{C}$ and all natural numbers $n \geq 1$, we have $|z^n| = |z|^n$ and $\overline{z^n} = \overline{z}^n$.

If you use the properties of $\overline{z}$ that we proved in class and you proved in Problem 2, then the solutions to the first two parts of the next problem should be very short.

Problem 3(a). Let $r \in \mathbb{R}$ and $z \in \mathbb{C}$. Prove that $\overline{rz} = r \overline{z}$.

3(b). Let $p(z)$ be a polynomial with real coefficients. That is, $p(z)$ looks like

$$p(z) = r_0 + r_1 z + r_2 z^2 + \cdots + r_n z^n$$

with $r_0, r_1, \ldots, r_n \in \mathbb{R}$. Prove that $\overline{p(z)} = p(\overline{z})$.

3(c). Use 3(b) to explain why the non-real zeros of $p(z)$ must occur in conjugate pairs. That is, explain why if $p(z) = 0$ and $z$ is not real, then $p(\overline{z}) = 0$ as well.

One interesting feature of working with the exponential function in $\mathbb{C}$ is that it makes some trig identities very easy to prove. The next two problems will give you three examples of this phenomenon.

Problem 4(a). Prove that if $\theta$ is real, then

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

*Hint.* Write $e^{i\theta}$ and $e^{-i\theta}$ in terms of their real and imaginary parts and then add them. How are $\cos \theta$ and $\cos -\theta$ related?

4(b). Use 4(a) to prove that $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$.

*Hint.* Square both sides of the equation in 4(a), then use 4(a) again when you are simplifying.
Problem 5(a). Prove that if $\theta$ is real and $n$ is a positive integer then 

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

*Hint.* For what complex value $z$ is $e^z$ equal to $\cos \theta + i \sin \theta$? Once you find $z$, use the fact that $(e^z)^n = e^{nz}$.

5(b). Use the case of $n = 3$ in 5(a) to prove that

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta \quad \text{and} \quad \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

*Hint.* The case of $n = 3$ in 5(a) tells you that $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$. Multiply out the lefthand side and then compare the real and imaginary parts of the two sides of the equation.

Practice Problems

Exercises 9.6: 2, 3, 4(a)(b)(c), 5(a)(b)(c)

Exercises 9.10: 1, 3, 4