Math 2142 Homework 2: Due Friday February 6

**Problem 1.** Consider the behavior of the numerators and denominators to find the following one sided limits

\[
\lim_{x \to 0^+} \frac{3x - 4}{\ln(x)} \quad \lim_{x \to -3^+} \frac{2x + 7}{x^2 - 9} \quad \lim_{x \to -3^-} \frac{2x + 7}{x^2 - 9}
\]

**Problem 2.** Prove that for any polynomial \( p(x) \) we have

\[
\lim_{x \to \infty} \frac{p(x)}{e^x} = 0
\]

*Hint.* Prove this statement by induction on the degree of \( p(x) \) using L'Hopital's Rule. This problem shows formally that the exponential function \( e^x \) grows much faster than any polynomial.

**Problem 3.** Prove that for any \( n \geq 2 \), the log function \( \ln(x) \) grows slower than \( \sqrt[n]{x} \) by proving that

\[
\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0
\]

(You do not need induction here. You can calculate this limit directly using L'Hopital’s Rule.)

**Problem 4.** Find the following limits. You are welcome (and encouraged) to use L’Hopital’s Rule. However, you should be careful when you apply L’Hopital’s Rule that the quotient has the correct form!

4(a).

\[
\lim_{x \to \infty} \frac{3x^2 - x + 7}{e^x}
\]

4(b).

\[
\lim_{x \to \infty} \frac{(\ln(x))^2}{\sqrt{x}}
\]

4(c).

\[
\lim_{x \to \infty} \frac{\sin(x) - x}{x^3}
\]

4(d).

\[
\lim_{x \to 1^+} \frac{\sqrt{x} - 1 + \sqrt{x} - 1}{\sqrt{x^2} - 1}
\]

4(e).

\[
\lim_{x \to 0^+} \sqrt{x} \ln(x)
\]

4(f).

\[
\lim_{x \to 0^+} \sin(x) \ln(x)
\]
4(g). \[ \lim_{x \to \infty} xe^{1/x} - x \]

4(h). \[ \lim_{x \to 0^+} x\sqrt{x} \]

4(i). \[ \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x \]

**Problem 5.** Determine whether the following integrals converge or diverge. If they converge, calculate their value.

\[ \int_0^3 \frac{1}{\sqrt{x}} \, dx \]

\[ \int_1^\infty \frac{\ln(x)}{x} \, dx \]

\[ \int_1^\infty \frac{\ln(x)}{x^2} \, dx \]

**Problem 6.** Recall that we showed in class that \( \int_1^\infty \frac{1}{x} \, dx = \infty \), which we can interpret as saying that the region under the graph of \( y = \frac{1}{x} \) with \( x \geq 1 \) has infinite area. However, if you look at the solid of revolution obtained by rotating this region around the \( x \)-axis, the formula for the volume of this solid is given by the integral

\[ \pi \int_1^\infty \frac{1}{x^2} \, dx \]

(This is the formula you obtain by using the disk method for determining the volume of the solid of revolution. You do not need to derive this formula.) Show that this integral converges by calculating its (finite) value.

The point of Problem 6 is that area and volume can behave very strangely when infinite integrals are involved. The region under the curve \( 1/x \) with \( x \geq 1 \) has infinite area and yet it still generates a solid with finite volume when rotated around the \( x \)-axis. Or in other words, the solid obtained by rotating this region around the \( x \)-axis has finite volume but has an infinite cross sectional area. Very strange!

**Practice Problems.** Here is a list of additional problems you can work on for more practice. You do not need to hand these in.

- Exercises 7.13: 1,2,4,5,7.