Math 2141 Homework 2 Solutions

Problem 1. Express the following sets as a single open, closed or half-open interval.

1(a). \([2, 4] \cup (1, 3) = (1, 4]\)
1(b). \([2, 4] \cap (1, 3) = [2, 3]\)
1(c). \(([0, 17] \cap [1, 4]) \cup [4, 7] = [0, 4) \cup [4, 7] = [0, 7]\)
1(d). \(([1, 12] - [-2, 3]) \cup (5, 14] = (3, 12] \cup (5, 14] = (3, 14]\)

Problem 2. For each of the following functions \(f : \mathbb{R} \to \mathbb{R}\), sketch the graph to determine if it is 1-to-1, onto, both or neither. If necessary, restrict the domain and/or codomain of the function so that it is 1-to-1 and onto.

2(a). \(f(x) = e^x\)
2(b). \(f(x) = (x - 3)^2 + 4\)
2(c). \(f(x) = x^3 + 1\)

Solution to 2(a). Looking at the graph

we see that \(f(x) = e^x\) is 1-to-1 but not onto as a function \(f : \mathbb{R} \to \mathbb{R}\). If we restrict the codomain to the interval \((0, \infty)\), then \(f : \mathbb{R} \to (0, \infty)\) is 1-to-1 and onto.

Solution to 2(b). Looking at the graph

we see that \(f(x) = (x - 3)^2 + 4\) is neither 1-to-1 nor onto as a function \(f : \mathbb{R} \to \mathbb{R}\). There are two natural restrictions on the domain to make the function 1-to-1: \((\infty, 3]\) and \([3, \infty)\). For either of these choices for the domain, restricting the codomain to the interval \([4, \infty)\) will make the function onto.
Solution to 2(c). Looking at the graph

we see that \( f(x) = x^3 + 1 \) is both 1-to-1 and onto.

Problem 3. Let \( f(x) = ax + b \) be the equation for a line where \( a \neq 0 \). The two parts of this problem ask you to prove that \( f(x) \) is 1-to-1 in two different ways.

3(a). Prove \( f(x) \) is 1-to-1 using our direct method of proof. That is, assume that \( r_0 \neq r_1 \) and show that \( f(r_0) \neq f(r_1) \). To help you get started, notice that \( f(r_0) = ar_0 + b \) and \( f(r_1) = ar_1 + b \).

3(b). Prove that \( f(x) \) is 1-to-1 by proving the contrapositive. That is, assume that \( f(r_0) = f(r_1) \) and show that \( r_0 = r_1 \).

Solution to 3(a). Let \( f(x) = ax + b \) with \( a \neq 0 \). To use a direct method of proof to show that \( f \) is 1-to-1, we assume that \( r_0 \neq r_1 \) are real numbers. We need to show that \( f(r_0) \neq f(r_1) \).

Starting with \( r_0 \neq r_1 \), we can multiply both sides of the inequality by \( a \) to get \( ar_0 \neq ar_1 \). (Notice that we are using the fact that \( a \neq 0 \) in this step.) Then adding \( b \) to both sides, we get that \( ar_0 + b \neq ar_1 + b \). Since \( f(r_0) = ar_0 + b \) and \( f(r_1) = ar_1 + b \), we have shown that \( f(r_0) \neq f(r_1) \) are required.

Solution to 3(b). To prove the same statement using the proof method of contraposition, we assume that \( f(r_0) = f(r_1) \) and we have to show that \( r_0 = r_1 \).

Since \( f(r_0) = ar_0 + b \) and \( f(r_1) = ar_1 + b \), our assumption that \( f(r_0) = f(r_1) \) means that \( ar_0 + b = ar_1 + b \). Subtracting \( b \) from both sides gives \( ar_0 = ar_1 \). Dividing by \( a \) (which is allowed because \( a \neq 0 \)) gives \( r_0 = r_1 \) which is what we needed to show.

Problem 4. Prove that \( f(x) = 3x + 2 \) is strictly increasing without using calculus.

Solution. We assume that \( r_0 < r_1 \) are real numbers and \( f(x) = 3x + 2 \). We need to show that \( f(r_0) < f(r_1) \).

Starting with \( r_0 < r_1 \), we can multiply both side by 3 to get \( 3r_0 < 3r_1 \). Adding 2 to both sides gives \( 3r_0 + 2 < 3r_1 + 2 \). Since \( f(r_0) = 3r_0 + 2 \) and \( f(r_1) = 3r_1 + 2 \), this means \( f(r_0) < f(r_1) \) which is what we needed to show.
**Problem 5.** Prove that if \( f : \mathbb{R} \to \mathbb{R} \) is strictly increasing, then \( f \) is 1-to-1.

**Solution.** I will give two slightly different ways to write this proof – either one is correct and it is useful for you to see both written versions.

**First proof.** Assume that \( f \) is strictly increasing and that \( r_0 \neq r_1 \). We show that \( f(r_0) \neq f(r_1) \). Since \( r_0 \neq r_1 \), we know that either \( r_0 < r_1 \) or \( r_1 < r_0 \). We can split our proof into two cases depending on whether \( r_0 < r_1 \) or \( r_1 < r_0 \). For each of the cases, we need to show that \( f(r_0) \neq f(r_1) \).

For the first case, suppose that \( r_0 < r_1 \). Because \( f \) is strictly increasing, this implies that \( f(r_0) < f(r_1) \), and so in particular, \( f(r_0) \neq f(r_1) \) as required.

For the second case, suppose that \( r_1 < r_0 \). Again, since \( f \) is strictly increasing, this implies that \( f(r_1) < f(r_0) \), and so in particular, \( f(r_0) \neq f(r_1) \).

**Second proof.** This proof is essentially the same as the first proof, but is written in a more concise manner. Assume that \( f \) is strictly increasing and we are given two different real numbers \( r_0 \) and \( r_1 \). We need to show that \( f(r_0) \neq f(r_1) \).

Since \( r_0 \) and \( r_1 \) are unequal, one must be less than the other. We can assume that we have labeled these numbers so that \( r_0 < r_1 \). Because \( f \) is strictly increasing, we have that \( f(r_0) < f(r_1) \), so in particular, \( f(r_0) \neq f(r_1) \) which is what we needed to show.

**Problem 6.** Let \( f(x) = ax + b \) be the equation of line with \( a > 0 \).

6(a). Prove that \( f(x) \) is strictly increasing without calculus. (Hint: Think about Problem 4)

6(b). Use Problems 5 and 6(a) to explain why \( f(x) \) is 1-to-1.

**Solution to 6(a).** Assume that \( r_0 < r_1 \). We need to show that \( f(r_0) < f(r_1) \). Because \( a > 0 \), we can multiply both sides of the inequality \( r_0 < r_1 \) by \( a \) without changing the direction of the inequality. Therefore, \( ar_0 < ar_1 \). Adding \( b \) to both sides gives \( ar_0 + b < ar_1 + b \). Since \( f(r_0) = ar_0 + b \) and \( f(r_1) = ar_1 + b \), we have that \( f(r_0) < f(r_1) \) as required.

**Solution to 6(b).** By Problem 6(a), \( f \) is strictly increasing which means (by Problem 5) that \( f \) is 1-to-1.

**Problem 7.** Prove that if \( k \) is a prime number, then \( \sqrt{k} \) is irrational.

**Solution.** We do this proof by contradiction. Assume that \( \sqrt{k} \) is rational. We write \( \sqrt{k} = p/q \) (with \( p, q \in \mathbb{N} \)) as a reduced fraction so that \( p \) and \( q \) do not have any common divisors.

Squaring both sides of \( \sqrt{k} = p/q \) gives \( k = p^2/q^2 \) which means \( kq^2 = p^2 \). Since \( k \) divides \( kq^2 \), \( k \) must also divide \( p^2 \). However, since \( k \) is prime, this means \( k \) divides \( p \) so we can write \( p = kn \) for some \( n \in \mathbb{N} \).

Plugging \( p = kn \) into \( kq^2 = p^2 \) gives \( kq^2 = k^2 n^2 \) which means \( q^2 = kn^2 \). Since \( k \) divides \( kn^2 \), \( k \) must divide \( q^2 \). Since \( k \) is prime, this means \( k \) divides \( q \). We have now shown that \( k \) divides both \( p \) and \( q \), which contradicts the fact that \( p \) and \( q \) have no common divisors.