Math 2141 Homework 1 Solutions

Problem 1. Explain why \( \{x \in \mathbb{N} \mid x \leq 2\} = \{x \in \mathbb{R} \mid x^3 - 3x^2 + 2x = 0\} \).

Solution. By definition, \( \{x \in \mathbb{N} \mid x \leq 2\} = \{0, 1, 2\} \). Since \( x^3 - 3x^2 + 2x = x(x - 1)(x - 2) \), we also have \( \{x \in \mathbb{R} \mid x^3 - 3x^2 + 2x = 0\} = \{0, 1, 2\} \). Therefore these two sets are equal.

Problem 2. Explain why \( \{x \in \mathbb{R} \mid x^3 - 2x^2 + x - 2\} \neq \{0, 1, 2\} \) and find the elements in

\( \{x \in \mathbb{R} \mid x^3 - 2x^2 + x = 2\} \cap \{0, 1, 2\} \).

Solution. Let \( A = \{x \in \mathbb{R} \mid x^3 - 2x^2 + x = 2\} \). If we plug \( x = 0 \) into \( x^3 - 2x^2 + x \) we get 0 and therefore \( x = 0 \) does not satisfy \( x^3 - 2x^2 + x = 2 \). This means that \( 0 \not\in A \) and hence \( A \neq \{0, 1, 2\} \).

Similarly, if we plug \( x = 1 \) into \( x^3 - 2x^2 + x \) we get 0 and therefore \( x = 1 \) does not satisfy \( x^3 - 2x^2 + x = 2 \). This means that \( 1 \not\in A \). However, if we plug \( x = 2 \) into \( x^3 - 2x^2 + x \) we get 2 and therefore \( x = 2 \) does satisfy \( x^3 - 2x^2 + x = 2 \). This means that \( 2 \in A \). Since \( 0, 1 \not\in A \) and \( 2 \in A \), we have \( \{x \in \mathbb{R} \mid x^3 - 2x^2 + x = 2\} \cap \{0, 1, 2\} = \{2\} \).

Problem 3. Prove that the following properties hold for all sets \( A, B \) and \( C \).

3(a). If \( A \subseteq B \) and \( A \subseteq C \), then \( A \subseteq B \cap C \).

3(b). If \( A \subseteq B \cup C \) and \( A \cap B = \emptyset \), then \( A \subseteq C \).

3(c). \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \).

3(d). \( A - (B \cup C) = (A - B) \cap (A - C) \).

Solution to 3(a). Assume that \( A \subseteq B \) and \( A \subseteq C \). We need to show that \( A \subseteq B \cap C \). That is, we need to show that if \( a \in A \), then \( a \in B \cap C \). Therefore, we assume that \( a \in A \) and show that \( a \in B \cap C \).

Since \( a \in A \) and \( A \subseteq B \), we know that \( a \in B \). Since \( a \in A \) and \( A \subseteq C \), we know that \( a \in C \). Therefore \( a \in B \) and \( a \in C \), which means \( a \in B \cap C \) as required.

Solution to 3(b). Assume that \( A \subseteq B \cup C \) and \( A \cap B = \emptyset \). We need to show that \( A \subseteq C \). That is, we need to show that if \( a \in A \), then \( a \in C \). Therefore, we assume that \( a \in A \) and we show that \( a \in C \).

Since \( a \in A \) and \( A \subseteq B \cup C \), we know that \( a \in B \cup C \). This means that either \( a \in B \) or \( a \in C \). However, \( A \cap B = \emptyset \) so there are no elements which are in both \( A \) and \( B \). Since \( a \in A \), we know that \( a \not\in B \). Collecting our facts together, we know that either \( a \in B \) or \( a \in C \), and that \( a \not\in B \). Therefore, it must be that \( a \in C \) which is what we needed to show.
Solution to 3(c). For this problem, we split the equality into the two subset relations and show each subset relation holds.

First, we show that

\[ A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C). \]

Assume that \( x \in A \cup (B \cap C) \) and we show that \( x \in (A \cup B) \cap (A \cup C) \). Since \( x \in A \cup (B \cap C) \), we know that either \( x \in A \) or \( x \in B \cap C \). We split into two cases depending on which of these cases holds.

For the first case, suppose that \( x \in A \). Since \( x \in A \), we know that \( x \in A \cup B \) and \( x \in A \cup C \). Therefore, \( x \in (A \cup B) \cap (A \cup C) \) as required.

For the second case, suppose that \( x \in B \cap C \). This means that \( x \in B \) and \( x \in C \). Since \( x \in B \), we know that \( x \in A \cup B \). Since \( x \in C \), we know that \( x \in A \cup C \). Therefore, \( x \) is in both \( A \cup B \) and \( A \cup C \), so \( x \in (A \cup B) \cap (A \cup C) \).

Second, we show that

\[ (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C). \]

Assume that \( x \in (A \cup B) \cap (A \cup C) \) and we show that \( x \in A \cup (B \cap C) \). Since \( x \in (A \cup B) \cap (A \cup C) \), we know that \( x \in A \cup B \) and \( x \in A \cup C \). We split into two cases depending on whether \( x \in A \) or \( x \notin A \).

For the first case, suppose that \( x \in A \). Since \( x \in A \), we know that \( x \in A \cup (B \cap C) \) as required.

For the second case, suppose that \( x \notin A \). Recall that we know \( x \in A \cup B \) and \( x \in A \cup C \). Since \( x \in A \cup B \), we know that either \( x \in A \) or \( x \in B \). But, \( x \notin A \), so it must be that \( x \in B \).

Similarly, since \( x \in A \cup C \), we know that either \( x \in A \) or \( x \in C \). But, \( x \notin A \), so it must be that \( x \in C \). We have now shown that in this case, \( x \in B \) and \( x \in C \). This means that \( x \in B \cap C \) and hence \( x \in A \cup (B \cap C) \) as required.

Solution to 3(d). For this problem, we again break the equality into the two subset relations and show each of the subset relations holds.

First, we show that

\[ A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C). \]

Assume that \( x \in A \setminus (B \cup C) \) and we show that \( x \in (A \setminus B) \cap (A \setminus C) \). Since \( x \in A \setminus (B \cup C) \), we know that \( x \in A \) and \( x \notin B \cup C \). To say that \( x \notin B \cup C \) means that \( x \notin B \) and \( x \notin C \).

Therefore, we know that \( x \in A \), \( x \notin B \) and \( x \notin C \). Since \( x \in A \) and \( x \notin B \), we have \( x \in A \setminus B \). Since \( x \in A \) and \( x \notin C \), we have \( x \in A \setminus C \). Therefore, \( x \in (A \setminus B) \cap (A \setminus C) \).

Second, we show that

\[ (A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C). \]

Assume that \( x \in (A \setminus B) \cap (A \setminus C) \) and we show that \( x \in A \setminus (B \cup C) \). Since \( x \in (A \setminus B) \cap (A \setminus C) \), we know that \( x \in A \setminus B \) and \( x \in A \setminus C \). Since \( x \in A \setminus B \), we have \( x \in A \) and \( x \notin B \). Since \( x \in A \setminus C \), we have \( x \in A \) and \( x \notin C \). Collecting up our facts, we know that \( x \in A \), \( x \notin B \) and \( x \notin C \). Because \( x \notin B \) and \( x \notin C \), we get \( x \notin B \cup C \). Therefore, we know \( x \in A \) and \( x \notin B \cup C \) which means \( x \in A \setminus (B \cup C) \) as required.
Problem 4. Let $A$ and $B$ be sets such that $A \cap B = \emptyset$. What is $A - B$? Explain your answer.

Solution. I claim that $A - B = A$. To see why this equality is true, we explain each of the inclusions $A - B \subseteq A$ and $A \subseteq A - B$ separately.

First, we show that $A - B \subseteq A$. Assume that $a \in A - B$ and we show that $a \in A$. Since $a \in A - B$, we know that $a \in A$ and $a \notin B$. In particular, $a \in A$ which is what we needed to show.

Second, we show that $A \subseteq A - B$. Assume that $a \in A$ and we show that $a \in A - B$. Since $a \in A$ and $A \cap B = \emptyset$, we know that $a \notin B$. However, since $a \in A$ and $a \notin B$, this means that $a \in A - B$ which is what we needed to show.

Problem 5. Let $A$, $B$, $C$ and $D$ be sets. Prove that if $A \subseteq B$ and $C \subseteq D$, then $A - D \subseteq B - C$.

Solution. Assume that $A \subseteq B$ and $C \subseteq D$. To show $A - D \subseteq B - C$, we assume that $x \in A - D$ and show that $x \in B - C$.

Because $x \in A - D$, we know that $x \in A$ and $x \notin D$. Since $x \in A$ and $A \subseteq B$, we get that $x \in B$. Since $x \notin D$ and $C \subseteq D$, we have $x \notin C$. Therefore, we know $x \in B$ and $x \notin C$, which means $x \in B - C$.

Problem 6(a). Draw two Venn diagrams and sketch the regions described by $A - (B - C)$ and by $(A - B) \cup C$.

6(b). Give an example of sets $A$, $B$ and $C$ for which $A - (B - C) \neq (A - B) \cup C$.

Solution to 6(a). Here are the Venn diagrams.

![Venn Diagrams](image)

Solution to 6(b). From the Venn diagrams, we see that $(A - B) \cup C$ includes the region $C - A$ but the region $C - A$ is not included in $A - (B - C)$. So, we just need an example of sets $A$, $B$ and $C$ for which $C - A$ is not empty. Let $A = \{0, 1, 2\}$, $B = \{1, 2, 3\}$ and $C = \{2, 3, 4\}$.

$$A - (B - C) = \{0, 1, 2\} - (\{1, 2, 3\} - \{2, 3, 4\}) = \{0, 1, 2\} - \{1\} = \{0, 2\}$$

$$(A - B) \cup C = (\{0, 1, 2\} - \{1, 2, 3\}) \cup \{2, 3, 4\} = \{0\} \cup \{2, 3, 4\} = \{0, 2, 3, 4\}$$