Separating the Degree Spectra of Structures

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In computable model theory, mathematical structures are studied on the basis of their computability or computational complexity. The degree spectrum $\mathrm{DgSp}(\mathfrak{A})$ of a countable structure $\mathfrak{A}$ is one way to measure the computability of the structure. Given various classes of countable structures, such as linear orders, groups, and graphs, we separate two classes $\mathcal{K}_1$ and $\mathcal{K}_2$ in the following way: we say that $\mathcal{K}_1$ is *distinguished from $\mathcal{K}_2$ with respect to degree spectrum* if there is an $\mathfrak{A} \in \mathcal{K}_1$ such that for all $\mathfrak{B} \in \mathcal{K}_2$, $\mathrm{DgSp}(\mathfrak{A}) \neq \mathrm{DgSp}(\mathfrak{B})$. In the dissertation, we will investigate this separation idea. We look at specific choices for $\mathcal{K}_1$ and $\mathcal{K}_2$—for example, we show that linear orders are distinguished from finite-components graphs, equivalence structures, rank-1 torsion-free abelian groups, and daisy graphs with respect to degree spectrum. Out of these proofs, there comes a general pattern for the kinds of structures from which linear orders are distinguished with respect to degree spectrum. In the future, we may also replace linear orders with possibly more general kinds of structures.

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Chapter 1

Introduction

1.1 Computable model theory

This dissertation is in computable model theory. We will assume that the reader is familiar with the basic notions of computable model theory as presented in Ash & Knight [2] and Downey [12].

In this dissertation, all languages are computable and all structures are countable. Specifically, every structure $\mathfrak{A}$ is assumed to have domain (or universe) $|\mathfrak{A}| = \mathbb{N}$. We drop mention of the specific language in use (e.g., we often write "structure" instead of "$\mathcal{L}$-structure") either when the mention of it is unimportant or when the language is clear from context.

Now in order for us to study computable model theory, we first need a way to measure the computational complexity of structures. The *degree* of a structure measures such complexity and is defined as follows.

Definition 1.1.1. Let $\mathcal{L}$ be a computable language, and let $\mathfrak{A}$ be an $\mathcal{L}$-structure with domain $|\mathfrak{A}| = \mathbb{N}$. The *degree* of $\mathfrak{A}$, denoted $\deg(\mathfrak{A})$, is the Turing degree of the set

$$D(\mathfrak{A}) = \{\varphi(\overline{a}) \mid \varphi(\overline{x}) \text{ is an atomic or negated atomic } \mathcal{L}\text{-formula and } \mathfrak{A} \models \varphi(\overline{a})\}.$$

We call $D(\mathfrak{A})$ the *atomic diagram* of $\mathfrak{A}$. In this definition, and throughout this dissertation, we equate formulas with their Gödel numbers.

One tool that captures the entire range of possible degrees (or codings) of a structure $\mathfrak{A}$ is the *degree spectrum* of $\mathfrak{A}$, defined as follows.

Definition 1.1.2. Let $\mathfrak{A}$ be a countable structure. The *degree spectrum* (or *spectrum*) of $\mathfrak{A}$, denoted $\mathrm{DgSp}(\mathfrak{A})$, is the set

$$\mathrm{DgSp}(\mathfrak{A}) = \{\deg(\mathfrak{B}) \mid \mathfrak{B} \cong \mathfrak{A}\}.$$

As a quick note on terminology, an *isomorphic copy*, a *copy*, and a *presentation* of $\mathfrak{A}$ will all mean the same thing: a structure $\mathfrak{B}$ isomorphic to $\mathfrak{A}$. For a Turing degree $\mathbf{d}$, a $\mathbf{d}$-*copy* or a $\mathbf{d}$-*presentation* of $\mathfrak{A}$ is just a structure $\mathfrak{B} \cong \mathfrak{A}$ such that $\deg(\mathfrak{B}) = \mathbf{d}$.

For any structure that we discuss, we can assume its degree spectrum is closed upwards. Indeed, Julia Knight shows in [23] that degree spectra are closed upwards for *nontrivial* structures:

Theorem 1.1.3 (Knight)**.** *Let $\mathfrak{A}$ be a structure in a relational language. Then exactly one of the following holds:*

(1) If $\mathbf{d} \geq \deg(\mathfrak{A})$, then $\mathbf{d} = \deg(\mathfrak{B})$ for some $\mathfrak{B} \cong \mathfrak{A}$.

(2) There is a finite $S \subseteq \mathbb{N}$ such that all permutations of $\mathbb{N}$ that fix S are automorphisms of $\mathfrak{A}$.

In the intro chapter of Hirschfeldt, Khoussainov, Shore, and Slinko [20], we see some theorems concerning degree spectra, two of which we state below. The first theorem is a restatement of Theorem 1.1.3, only now we say that a structure $\mathfrak{A}$ is *trivial* if it satisfies condition (2) above. The second theorem tells of a structure whose degree spectrum contains all degrees but $\mathbf{0} = \deg_T(\emptyset)$.

Theorem 1.1.4 (Thm 1.19 in [20], Knight)**.** *If $\mathfrak{A}$ is not trivial, then $\mathrm{DgSp}(\mathfrak{A})$ is closed upward.*

Theorem 1.1.5 (Thm 1.20 in [20], Slaman; Wehner)**.** *There is a structure $\mathfrak{A}$ such that $\mathrm{DgSp}(\mathfrak{A}) = \mathbf{D} - \{\mathbf{0}\}$, where $\mathbf{D}$ is the set of all Turing degrees.*

By a *class* of structures, we mean the collection of all structures that model a fixed theory. Now, given two distinct classes of structures, we wish to compare them based on the complexity of their structures, so we'll ask questions about the structures' degree spectra. One way to compare the degree spectra of structures from one class and the degree spectra of structures in another class is to answer the question, "Given two classes of structures $\mathcal{K}_1$ and $\mathcal{K}_2$, is there a structure $\mathfrak{A} \in \mathcal{K}_1$ such that for *every* structure $\mathfrak{B} \in \mathcal{K}_2$, $\mathrm{DgSp}(\mathfrak{A}) \neq \mathrm{DgSp}(\mathfrak{B})$?" Questions of this form are the driving force of this dissertation.

1.2 Results of Richter

Implicit in our use of degree *spectra*, when comparing two structures, is the fact that the *degree* of a structure alone is not enough. Indeed, $\deg(\mathfrak{A})$ is not isomorphically invariant—sometimes $\deg(\mathfrak{A}) \neq \deg(\hat{\mathfrak{A}})$ even when $\mathfrak{A} \cong \hat{\mathfrak{A}}$. Linda Jean Richter partially remedies this problem in her thesis [28]. Indeed, we can often pin down one particular degree for all the isomorphic copies of a structure. If $\mathrm{DgSp}(\mathfrak{A})$ has a least element, then we define the *degree of the isomorphism class of* $\mathfrak{A}$ to be that least degree. To give a little background in the chapters to come, we'll now present some of Richter's results from [28] that explore degrees of isomorphism classes.

Definition 1.2.1. Given a structure $\mathfrak{A}$, a finite structure $\mathfrak{C}$, and an embedding $f : \mathfrak{C} \hookrightarrow \mathfrak{A}$, define the class

$$\begin{aligned} A_{\mathfrak{C},f} = \{\mathfrak{D} \mid \mathfrak{D} &\text{ is a finite structure extending } \mathfrak{C} \\ &\text{and embeddable in } \mathfrak{A} \text{ by a map extending } f\}. \end{aligned}$$

The structure $\mathfrak{A}$ is said to satisfy the *Computable Embeddability Condition (CEC)* iff for all finite $\mathfrak{C}$ embeddable in $\mathfrak{A}$ and for all functions f embedding $\mathfrak{C}$ into $\mathfrak{A}$, $A_{\mathfrak{C},f}$ is computable.

Theorem 1.2.2 (Richter). *For any countable structure $\mathfrak{A}$ which satisfies the CEC, there is an isomorphic structure $\mathfrak{B}$ such that $\{\deg(\mathfrak{A}), \deg(\mathfrak{B})\}$ is a minimal pair (i.e., if $\mathbf{c} \leq \deg(\mathfrak{A})$ and $\mathbf{c} \leq \deg(\mathfrak{B})$, then $\mathbf{c} = \mathbf{0}$).*

Richter showed that countable linear orders satisfy the CEC and hence obtained the following theorem.

Theorem 1.2.3 (Richter)**.** *For any countable linear order $\mathfrak{L}$, if* $\mathrm{DgSp}(\mathfrak{L})$ *has a least element, then that element is* $\mathbf{0}$.

The following theorem of Richter shows the conditions in which a sequence of special finite structures can be combined to form a structure whose spectrum can have any least element of our choosing.

Theorem 1.2.4 (Richter)**.** *Let $\mathcal{K}$ be a class of structures. If*

1. *there is a computable sequence $\mathfrak{A}_0, \mathfrak{A}_1, \ldots$ of finite structures such that $\mathfrak{A}_i \not\hookrightarrow \mathfrak{A}_j$ for all $i \neq j$, and*
2. *for each $S \subseteq \mathbb{N}$, there is a structure $\mathfrak{A}_S$ such that:*
   - *(a) $\mathfrak{A}_S \in \mathcal{K}$,*
   - *(b) $D(\mathfrak{A}_S) \leq_T S$, and*
   - *(c) $\mathfrak{A}_i \hookrightarrow \mathfrak{A}_S$ iff $i \in S$,*

then, for any degree $\mathbf{d}$, there is an $\mathfrak{A} \in \mathcal{K}$ such that $\mathbf{d} = \min \mathrm{DgSp}(\mathfrak{A})$.

For a concrete example of this, we can apply the theorem to *finite-component graphs.*

Definition 1.2.5. In the following definitions, we assume graphs to be directed graphs.

(1) A graph is *connected* if every pair of vertices is connected by a sequence of edges.

(2) The *components* of a graph are its maximal connected subgraphs.

(3) A *finite-component graph* is a graph with only finite components.

Corollary 1.2.6. *For every degree* $\mathbf{d}$*, there is a finite-component graph* $\mathfrak{G}_{\mathbf{d}}$ *such that* $\mathrm{DgSp}(\mathfrak{G}_{\mathbf{d}})$ *has least element* $\mathbf{d}$.

Proof. Let $\mathcal{K}$ be the class of finite-component graphs, and let $\mathfrak{A}_i$ be a cycle of size $i+2$ (i.e., a graph whose set of edges has the form

$$\{(a_0, a_1), (a_1, a_2), \ldots, (a_i, a_{i+1}), (a_{i+1}, a_0)\}.)$$

For every $S \subseteq \mathbb{N}$, let $\mathfrak{A}_S$ be the disjoint union of cycles $\mathfrak{A}_i$ for all $i \in S$. By Theorem 1.2.4, our desired $\mathfrak{G}_{\mathbf{d}}$ exists. (Notice the reason that (b) holds. The set of edges $E^{\mathfrak{A}_S}$ of $\mathfrak{A}_S$ is c.e. in S. So given the question, "Is it the case that $vE^{\mathfrak{A}_S}u$?" for two vertices v and u, first determine whether or not v and u are in the same cycle of $\mathfrak{A}_S$ by listing $E^{\mathfrak{A}_S}$ and eventually finding all the vertices of the one or two cycles needed. If v and u are in separate cycles or if they are in the same cycle and are not connected by an edge, we output "no," but otherwise we output "yes.") $\square$

Now we have a specific example as a positive answer to our main question, "Given two classes of structures $\mathcal{K}_1$ and $\mathcal{K}_2$, is there a structure $\mathfrak{A} \in \mathcal{K}_1$ such that for *every* structure $\mathfrak{B} \in \mathcal{K}_2$, $\mathrm{DgSp}(\mathfrak{A}) \neq \mathrm{DgSp}(\mathfrak{B})$?"

Theorem 1.2.7. *There is a finite-component graph* $\mathfrak{G}$ *such that* $\mathrm{DgSp}(\mathfrak{G}) \neq \mathrm{DgSp}(\mathfrak{L})$ *for all linear orders* $\mathfrak{L}$*.*

Proof. Fix a linear order $\mathfrak{L}$, and let $\mathfrak{G}_{\mathbf{d}}$ be as in Corollary 1.2.6 for $\mathbf{d} = \mathbf{0}' = \deg_T(\emptyset')$, where $\emptyset' = \{e \mid \varphi_e(e)\downarrow\}$. Then $\mathbf{0} \notin \mathrm{DgSp}(\mathfrak{G}_{\mathbf{d}})$, which has a least element, $\mathbf{0}'$. But by Theorem 1.2.3 it's either the case that $\mathbf{0} \in \mathrm{DgSp}(\mathfrak{L})$ or $\mathrm{DgSp}(\mathfrak{L})$ has no least element. So $\mathrm{DgSp}(\mathfrak{L}) \neq \mathrm{DgSp}(\mathfrak{G}_{\mathbf{d}})$. □

As another note on directed graphs, although the proof of Theorem 1.1.5 originally used more exotic structures, it can be modified to show that there is a *directed graph* whose degree spectrum is all the degrees but $\mathbf{0}$. Now in light of our above result, this highlights an open question: Can a linear order achieve this spectrum? In Chapter 2, we will prove the reverse separation of degree spectra for finite-component graphs and linear orders. Specifically, we will show that there is a linear order whose degree spectrum cannot be realized by any finite-component graph.

Rank-1 torsion-free abelian groups give a second example of this kind of theorem.

Definition 1.2.1. (1) A group $\mathfrak{G}$ is said to have *torsion* if there is an element of finite order, and $\mathfrak{G}$ is *torsion-free* otherwise.

(2) The *rank* of a group $\mathfrak{G}$ is the least n such that $\mathfrak{G}$ is isomorphic to a subgroup of $\mathbb{Q}^n$. So a rank-1 group is a group isomorphic to a subgroup of $\mathbb{Q}$.

In Coles, Downey, and Slaman [7], we have a proof of the following.

Theorem 1.2.8 (Downey). *For every degree* $\mathbf{d}$*, there is a rank-1 torsion-free abelian group* $\mathfrak{G}_{\mathbf{d}}$ *such that* $\mathrm{DgSp}(\mathfrak{G}_{\mathbf{d}})$ *has least element* $\mathbf{d}$.

So by arguing as in Theorem 1.2.7, we can make another separation of degree spectra:

Theorem 1.2.9. *There is a rank-1 torsion-free abelian group* $\mathfrak{G}$ *such that* $\mathrm{DgSp}(\mathfrak{G}) \neq \mathrm{DgSp}(\mathfrak{L})$ *for all linear orders* $\mathfrak{L}$.

In Chapter 2, we will prove the reverse separation of degree spectra for rank-1 torsion-free abelian groups and linear orders. Specifically, we will show that there is a linear order whose degree spectrum cannot be realized by any rank-1 torsion-free abelian group.

1.3 A result of Knight

There are some degree spectra without a least element. For example, as seen from Theorem 1.2.3, if a linear order $\mathfrak{L}$ has no $\mathbf{0}$-presentation, then $\mathrm{DgSp}(\mathfrak{L})$ has no least element. In response to this, instead of finding the least of the *degrees*, Carl Jockusch suggested finding the least of the *jump(s)* of the degrees. Recall that for a set X, the *jump* of X is the set $X' = \{e \mid \varphi_e^X(e)\downarrow\}$, and the n^{th} *jump* of X for $n > 0$ is $X^{(n)} = (X^{(n-1)})'$, where $X^{(0)} = X$. Also, if $\mathbf{x} = \deg_T(X)$, then we write $\mathbf{x}^{(n)} = \deg_T(X^{(n)})$, $\mathbf{x}' = \mathbf{x}^{(1)}$, $\mathbf{x}'' = \mathbf{x}^{(2)}$, and $\mathbf{x}''' = \mathbf{x}^{(3)}$.

Definition 1.3.1. A structure $\mathfrak{A}$ has n^{th} *jump degree* $\mathbf{d}$ if

$$\mathbf{d} = \min\{\deg(\mathfrak{B})^{(n)} \mid \mathfrak{B} \cong \mathfrak{A}\}.$$

Knight [23] shows the following.

Theorem 1.3.2 (Knight)**.** *If a linear order $\mathfrak{L}$ has a jump degree (i.e., a 1st jump degree), then that degree is $\mathbf{0}'$.*

So we see that linear orders with no presentation of low degree (a subclass of the linear orders with no $\mathbf{0}$-presentation) will be without jump degree. Also, this leads to a certain separation of degree spectra for *equivalence structures* and linear orders:

> There is an equivalence structure $\mathfrak{E}$ such that $\mathrm{DgSp}(\mathfrak{E}) \neq \mathrm{DgSp}(\mathfrak{L})$ for any linear order $\mathfrak{L}$.

Definition 1.3.3. An *equivalence structure* $\mathfrak{E}$ is a structure consisting of one relation $\sim_{\mathfrak{E}}$ that is an equivalence relation (i.e., for all $x, y, z \in |\mathfrak{E}|$, we have $x \sim_{\mathfrak{E}} x$, $x \sim_{\mathfrak{E}} y \implies y \sim_{\mathfrak{E}} x$, and $x \sim_{\mathfrak{E}} y \wedge y \sim_{\mathfrak{E}} z \implies x \sim_{\mathfrak{E}} z$). For each $x \in |\mathfrak{E}|$, let $[x]_{\mathfrak{E}}$ denote the *equivalence class* of $\mathfrak{E}$ that contains x.

Theorem 1.3.4. *Let $\mathbf{d} \geq \mathbf{0}'$. There is an equivalence structure with jump degree $\mathbf{d}$.*

Proof. Fix $D \in \mathbf{d}$ such that $0 \notin D$, without loss of generality. Let $\mathfrak{E} = (\mathbb{N}, \sim_{\mathfrak{E}})$ be the equivalence structure with

- one class of size $2n$ for each $n \in D$,
- one class of size $2n+1$ for each $n \notin D$, and
- infinitely many classes of infinite size.

To show that $\mathfrak{E}$ has jump degree $\mathbf{d}$, it suffices to show that (1) for any $\mathfrak{F} \cong \mathfrak{E}$, $D \leq_T D(\mathfrak{F})'$ and that (2) there is an $\mathfrak{F} \cong \mathfrak{E}$ such that $D(\mathfrak{F})' \leq_T D$.

(1): Let $\mathfrak{F} = (\mathbb{N}, \sim_{\mathfrak{F}})$ be isomorphic to $\mathfrak{E}$. For any s, let $\mathfrak{F}_s = (\{0, 1, \ldots, s\}, \sim_{\mathfrak{F}})$. For each $x \leq s$, $D(\mathfrak{F})'$ can answer

$$(\exists y > s) x \sim_{\mathfrak{F}} y.$$

If "No," then $[x]_{\mathfrak{F}_s} = [x]_{\mathfrak{F}}$, and so $D(\mathfrak{F})'$ knows $|[x]_{\mathfrak{F}}|$. To compute D, at each stage s, $D(\mathfrak{F})'$ asks for each $x \leq s$ whether $[x]_{\mathfrak{F}_s} = [x]_{\mathfrak{F}}$ and, if they are equal, calculates $|[x]_{\mathfrak{F}_s}|$. Eventually $D(\mathfrak{F})'$ finds a class of size $2n$ or $2n+1$ and can thus correctly determine whether $n \in D$.

(2) By the Friedberg Jump Inversion Theorem (i.e., Thm VI.3.1 in Soare [30], called there the Friedberg Completeness Criterion), fix C such that $C' \equiv_T D$. We show that there is an $\mathfrak{F} \cong \mathfrak{E}$ such that $D(\mathfrak{F}) \leq_T C$ (so that $D(F)' \leq_T C' \equiv_T D$). Since $D \leq_T C'$, there is a C-computable function $f(x, s)$ such that

$$x \in D \iff \lim_{s \to \infty} f(x, s) = 1 \quad \text{and} \quad x \notin D \iff \lim_{s \to \infty} f(x, s) = 0.$$

Build $\mathfrak{F}$ in stages:

At stage s, for each $x \leq s$, calculate $f(x,s)$ using C. If $f(x,s) = 1$ and there is currently no $\mathfrak{F}$-class of size $2x$, then add one. If there is an $\mathfrak{F}$-class of size $2x+1$, then turn it into an infinite class. If $f(x,s) = 0$ and there is no $\mathfrak{F}$-class of size $2x+1$, then add one. If there is an $\mathfrak{F}$-class of size $2x$, then turn it into an infinite class. Also, add one new infinite class at each stage. $\square$

So as mentioned we obtain the following separation of degree spectra.

Theorem 1.3.5. *There is an equivalence structure* $\mathfrak{E}$ *such that* $\mathrm{DgSp}(\mathfrak{E}) \neq \mathrm{DgSp}(\mathfrak{L})$ *for any linear order* $\mathfrak{L}$*.*

Proof. Let $\mathfrak{E}$ be an equivalence structure with jump degree $> \mathbf{0}'$. Let $\mathfrak{L}$ be a linear order. By Theorem 1.3.2, $\mathfrak{L}$ can only have jump degree $\mathbf{0}'$. Therefore, $\{\deg(\mathfrak{B})' \mid \mathfrak{B} \cong \mathfrak{E}\} \neq \{\deg(\mathfrak{B})' \mid \mathfrak{B} \cong \mathfrak{L}\}$, so $\mathrm{DgSp}(\mathfrak{E}) \neq \mathrm{DgSp}(\mathfrak{L})$. $\square$

1.4 Summary of results

Given two classes of structures, $\mathcal{K}_1$ and $\mathcal{K}_2$, we say that $\mathcal{K}_1$ is *distinguished from* $\mathcal{K}_2$ *with respect to degree spectrum* if there is an $\mathfrak{A} \in \mathcal{K}_1$ such that for every $\mathfrak{B} \in \mathcal{K}_2$, $\mathrm{DgSp}(\mathfrak{A}) \neq \mathrm{DgSp}(\mathfrak{B})$. From known results in computable model theory, we have seen some classes of structures that are distinguished from linear orders with respect to degree spectrum. In the coming chapters, we will reverse the question and show that linear orders are distinguished from the following classes:

- finite-component graphs,
- equivalence structures,
- rank-1 torsion-free abelian groups, and
- daisy graphs.

Although we will wait until Chapter 3 for the definition of daisy graphs, we will note here that there is a significant difference between the method we use for the first three of these classes and the method we use for daisy graphs. Indeed, the first three of these classes have canonical codes in the form of *sets*, whereas daisy graph have canonical codes in the form of *families of sets*. Chapter 2 is devoted to canonical codes that are *sets*. Section 2.1 extends the results of Knight ultimately linking degree spectra with sets. Sections 2.2-2.4 then apply this method to finite-component graphs, equivalence structures, and rank-1 torsion-free abelian groups. Chapter 3 is devoted to canonical codes that are *families of sets*. Section 3.1 takes a step beyond the methods of Section 2.1 to now link degree spectra with families of sets. And Section 3.2 applies this to daisy graph and in fact shows a general setup for the potential of more applications in the future.

Chapter 2

Extending Knight's result

2.1 The extension

We wish to find more examples that separate the degree spectra of structures in the sense of Chapter 1. And in fact, by first extending a theorem of Knight, we'll find examples of the specific form, "There is a linear order whose degree spectrum is not that of any ___(name of other structure here)___." Now Theorem 3.5 of Knight [23] says that if the jump of every copy of a linear order computes a fixed set $S \subseteq \mathbb{N}$, then in fact $\emptyset'$ computes S. In other words,

Theorem 2.1.1 (Knight [23])**.** *Let $\mathfrak{A}$ be a linear order, and suppose that $S \leq_T D(\mathfrak{B})'$ for all $\mathfrak{B} \cong \mathfrak{A}$. Then $S \leq_T \emptyset'$.*

To obtain the desired extending theorem, we have the following Σ^0_1 version of Theorem 2.1.1, and the rest of this section is devoted to its proof:

> **(Theorem 2.1.14.)** If $\mathfrak{A}$ is a linear order and $S \subseteq \mathbb{N}$ such that $S \in \Sigma_1^{D(\mathfrak{B})'}$ for all $\mathfrak{B} \cong \mathfrak{A}$, then $S \in \Sigma^0_2$.

This will be stated as Theorem 2.1.14 at the end of the section. Now to prove Theorem 2.1.14, we will use the forcing methods of §1 in [23] to build a particular copy of the linear order $\mathfrak{A}$ which will lead to the conclusion that $S \in \Sigma^0_2$. Specifically, fix a structure $\mathfrak{A}$. The set F of *forcing conditions* is the set of finite partial 1-1 functions $\mathbb{N} \to \mathbb{N}$ ordered by extension $\supseteq$. A chain of forcing conditions $p_0 \subseteq p_1 \subseteq \cdots$ yields a permutation $p = \bigcup_{i \in \mathbb{N}} p_i$ of $\mathbb{N}$ and the resulting model $\mathfrak{B}$ has its structure given by pulling the structure over from $\mathfrak{A}$ by p. That is, we make p an isomorphism between the structures.

We recall the definitions of the forcing statements used in [23] and add new statements needed for the proof of Theorem 2.1.14. Intuitively speaking, the notation $p \Vdash_{\mathfrak{A}} \psi$ will mean that $\mathfrak{A}$ is the structure being copied and p *forces* ψ, where ψ is a select statement about the atomic diagram of the copy $\mathfrak{B}$ of $\mathfrak{A}$ being built. Note that in the forcing language, D is used to denote $D(\mathfrak{B})$, and $D^{(m)}$ is used to denote $D(\mathfrak{B})^{(m)}$.

Definition 2.1.2. Let $p \Vdash_{\mathfrak{A}} \psi$ be defined as follows.

1. ψ is $k \in D$:

   $p \Vdash_{\mathfrak{A}} \psi \iff k$ is an *open* sentence (i.e., a sentence without quantifiers, recalling that we identify sentences with their Gödel numbers) with constants from ran(p) and the corresponding sentence with constants from dom(p) is in $D(\mathfrak{A})$.

   $p \Vdash_{\mathfrak{A}} \neg\psi \iff$ either k is not an open sentence in the language of $D(\mathfrak{B})$ or

k is such a sentence and $p \Vdash_{\mathfrak{A}} l \in D$, where l is the code for the negation of the formula coded by k.

2. ψ is $k \in D^{(m)}$ for some $m > 0$:

   $p \Vdash_{\mathfrak{A}} \psi \iff$ there are finite sets α and β such that $p \Vdash_{\mathfrak{A}} k \in D^{(m-1)}$ for all $k \in \alpha$, $p \Vdash_{\mathfrak{A}} k \notin D^{(m-1)}$ for all $k \in \beta$, and if $\alpha \subseteq X \subseteq \mathbb{N} - \beta$, then there is a halting computation of $\varphi_k^X(k)$ (using just this information about X).

   $p \Vdash_{\mathfrak{A}} \neg\psi \iff$ there is no $q \supseteq p$ such that $q \Vdash_{\mathfrak{A}} \psi$.

   Notice that forcing negation here is different than for $m = 0$.

3. ψ is $\varphi_e^{D^{(m)}}(k) = l$:

   $p \Vdash_{\mathfrak{A}} \psi \iff$ there are finite sets α, β such that $p \Vdash_{\mathfrak{A}} k \in D^{(m)}$ for all $k \in \alpha$, $p \Vdash_{\mathfrak{A}} k \notin D^{(m)}$ for all $k \in \beta$, and $\alpha \subseteq X \subseteq \mathbb{N} - \beta$ makes $\varphi_e^X(k) = l$ (i.e., some computation uses just this information about X).

   $p \Vdash_{\mathfrak{A}} \neg\psi \iff$ there is no $q \supseteq p$ such that $q \Vdash_{\mathfrak{A}} \psi$.

4. ψ is $\varphi_e^{D^{(m)}}(k)$ converges:

   $p \Vdash_{\mathfrak{A}} \psi \iff p \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(k) = l$ for some l.

   $p \Vdash_{\mathfrak{A}} \neg\psi$ (i.e., $p \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(k)$ diverges) $\iff$ there is no $q \supseteq p$ such that $q \Vdash_{\mathfrak{A}} \psi$.

After defining the needed forcing statements, §1 of [23] proceeds to describe how we actually build the desired copy $\mathfrak{B}$ of $\mathfrak{A}$. For every n, let $S(n)$ be the set of statements of the forms $k \in D^{(m)}$, $\varphi_e^{D^{(m)}}(k) = l$, and $\varphi_e^{D^{(m)}}(k)$ converges, for $m \leq n$. Let $p_0 \subseteq p_1 \subseteq \cdots$ be an *n-complete forcing sequence for $S(n)$* (i.e., a sequence such that for every $\psi \in S(n)$, there is an i such that $p_i \Vdash_{\mathfrak{A}} \psi$ or $p_i \Vdash_{\mathfrak{A}} \neg\psi$), and let $\pi = \bigcup_{i \in \mathbb{N}} p_i$. Then π is a permutation of $\mathbb{N}$, and $(p_i)_{i \in \mathbb{N}}$ determines a *generic* copy $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} \cong_\pi \mathfrak{B}$. Now we get the essential fact that $S(n)$-truth in $\mathfrak{B}$ is exactly forced, that is:

Fact 2.1.3 (Lemma 1.1 of Knight [23])**.** *For all $\psi \in S(n)$, ψ is true of $\mathfrak{B}$ iff for some i, $p_i \Vdash_{\mathfrak{A}} \psi$.*

To capture the notion that $S \leq_T D(\mathfrak{B})^{(m)}$ in the proof of Theorem 2.1.1, [23] continues the above list with an added informal forcing statement:

5. ψ is $\varphi_e^{D^{(m)}} = \chi_S$ for some $S \subseteq \mathbb{N}$:

   $p \Vdash_{\mathfrak{A}} \psi \iff$ for all k, there is some $q \supseteq p$ such that $q \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(k) = \chi_S(k)$, and for all $l \neq \chi_S(k)$, there is no $q \supseteq p$ such that $q \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(k) = l$.

This is not a formal forcing statement, since it is not preserved under extensions. Also, this motivates the definition of a new *quasi* forcing statement that will be useful to us. We use the notation "$\Diamond_{\mathfrak{A}}^p$" (echoing modal logic's *possibility* symbol $\Diamond$) instead of "$p \Vdash_{\mathfrak{A}}$" to indicate that the statement is not preserved under extensions (as formal forcing statements should be). Because our goal is to prove Theorem 2.1.14, we must capture the idea of S being $\Sigma_1^{D(\mathfrak{B})'}$ when building $\mathfrak{B}$, hence the following continuation of Definition 2.1.2.

Definition 2.1.4. Let $\Diamond_{\mathfrak{A}}^{p}\psi$ be defined as follows.

5. ψ is $W_e^{D^{(m)}} = S$ for some $S \subseteq \mathbb{N}$:

$$\Diamond_{\mathfrak{A}}^{p} W_e^{D^{(m)}} = S \iff S = \{k \mid (\exists q \supseteq p)(\exists l) q \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(k) = l\}.$$

As a final note on forcing before heading to our new results, we know that, for the basic statements ψ, it is easy to determine whether $p \Vdash_{\mathfrak{A}} \psi$. Specifically, we have the following fact.

Fact 2.1.5 (Lemma 1.2 of Knight [23])**.** *The relations $p \Vdash_{\mathfrak{A}} k \in D$ and $p \Vdash_{\mathfrak{A}} k \notin D$ are computable in $D(\mathfrak{A})$. For $m > 0$, the relation $p \Vdash_{\mathfrak{A}} k \in D^{(m)}$ is c.e. in $D(\mathfrak{A})^{(m-1)}$, and the relation $p \Vdash_{\mathfrak{A}} k \notin D^{(m)}$ is computable in $D(\mathfrak{A})^{(m)}$.*

In addition to forcing, we'll also use several properties of a family of equivalence relations $\sim_n$ between tuples.

Definition 2.1.6. For structures $\mathfrak{A}$ and $\mathfrak{B}$ and tuples $\bar{a} \in |\mathfrak{A}|$ and $\bar{b} \in |\mathfrak{B}|$, recursively define $\bar{a}_{\mathfrak{A}} \sim_n \bar{b}_{\mathfrak{B}}$ for all n as follows.

1. $\bar{a}_{\mathfrak{A}} \sim_0 \bar{b}_{\mathfrak{B}}$ if $\bar{a}$ and $\bar{b}$ satisfy the same open formulas, and

2. $\bar{a} \sim_{n+1} \bar{b}$ if for all $\bar{c} \in |\mathfrak{A}|$, there is a $\bar{d} \in |\mathfrak{B}|$ such that $\bar{a}, \bar{c}_{\mathfrak{A}} \sim_n \bar{b}, \bar{d}_{\mathfrak{B}}$, and for each $\bar{d} \in |\mathfrak{B}|$, there is a $\bar{c} \in |\mathfrak{A}|$ such that $\bar{a}, \bar{c}_{\mathfrak{A}} \sim_n \bar{b}, \bar{d}_{\mathfrak{B}}$.

We say that $\mathfrak{A} \sim_n \mathfrak{B}$ if $\varnothing_{\mathfrak{A}} \sim_n \varnothing_{\mathfrak{B}}$.

To get an intuitive understanding of $\sim_n$, fix $\mathfrak{A}$ and $\mathfrak{B}$ to be countable linear orders, and let $\bar{a} \in |\mathfrak{A}|$ and $\bar{b} \in |\mathfrak{B}|$. Then $\bar{a}_{\mathfrak{A}} \sim_0 \bar{b}_{\mathfrak{B}}$ if $\bar{a}$ and $\bar{b}$ are ordered in the same way, and $\overline{a}_{\mathfrak{A}} \sim_1 \overline{b}_{\mathfrak{B}}$ if $\overline{a}$ and $\overline{b}$ are ordered in the same way and partition $\mathfrak{A}$ and $\mathfrak{B}$ into intervals with the same number of elements. Our intuition starts to fade at the $n = 2$ level and beyond.

We are now prepared for the gauntlet of lemmas and facts that will yield the proof of Theorem 2.1.14. We start with the following Σ^0_1 versions of Lemmas 1.3 and 1.4 of [23].

Lemma 2.1.7. *Suppose that* $\Diamond^p_{\mathfrak{A}} W_e^{D^{(m)}} = S$*. Then* $S \in \Sigma_1^{D(\mathfrak{A})^{(m)}}$.

Proof. Consider the set of tuples of the form $(q, k, l, \alpha, \beta, C)$, where $q \supseteq p$, $q \Vdash_{\mathfrak{A}} j \in D^{(m)}$ for all $j \in \alpha$, $q \Vdash_{\mathfrak{A}} j \notin D^{(m)}$ for all $j \in \beta$, and C is a computation of $\varphi_e^X(k) = l$ using just the information that $\alpha \subseteq X \subseteq \mathbb{N} - \beta$. So by Fact 2.1.5, this set of tuples is c.e. in $D(\mathfrak{A})^{(m)}$. Then our hypothesis shows that for each k, $\chi_s(k) = 1$ if and only if there will be some tuple $(q, k, l, \alpha, \beta, C)$. $\square$

Lemma 2.1.8. *For all* $m \in \mathbb{N}$ *and* $S \subseteq \mathbb{N}$*, the following are equivalent:*

(1) $S \in \Sigma_1^{D(\mathfrak{B})^{(m)}}$ *for all* $\mathfrak{B} \cong \mathfrak{A}$.

(2) For some e *and some* $p \in F$, $\Diamond^p_{\mathfrak{A}} W_e^{D^{(m)}} = S$.

Proof. Suppose that (1) holds. For a contradiction, assume that for every p and e, $\Diamond^p_{\mathfrak{A}} W_e^{D^{(m)}} = S$ does not hold. Define a sequence of forcing conditions $p_0 \subseteq p_1 \subseteq \cdots$ such that the resulting model $\mathfrak{B}$ with $p : \mathfrak{A} \to \mathfrak{B}$ does not satisfy $S \in \Sigma_1^{D(\mathfrak{B})^{(m)}}$. Specifically, define this sequence to be m-complete for $S(m)$, interleaved with conditions that we will now explicitly define. Start with $p_0 = \emptyset$. Assume that we have p_e. By hypothesis, $\Diamond_{\mathfrak{A}}^{p_e} W_e^{D^{(m)}} = S$ does not hold. There are two cases. For the first case, suppose that there exists $k \in S$ such that for all $q \supseteq p_e$ and $l \in \mathbb{N}$, $q \nVdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(k) = l$. Then let $p_{e+1} = p_e$, as we already have that $k \notin W_e^{D(\mathfrak{B})^{(m)}}$. For the second case, suppose that there exist $k \notin S$, $q \supseteq p_e$, and $l \in \mathbb{N}$ such that $q \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(k) = l$. Then let $p_{e+1} = q$ and we have now forced that $k \in W_e^{D(\mathfrak{B})^{(m)}}$.

Next suppose that (2) holds. By Lemma 2.1.7, $S \in \Sigma_1^{D(\mathfrak{A})^{(m)}}$. Suppose $\mathfrak{A} \cong_f \mathfrak{B}$. If p takes $\overline{a}$ to $\overline{k}$, let q take $f(\overline{a})$ to $\overline{k}$. Then $\Diamond_{\mathfrak{B}}^{q} W_e^{D^{(m)}} = S$, and by Lemma 2.1.7, $S \in \Sigma_1^{D(\mathfrak{B})^{(m)}}$. □

The next fact comes directly from [23].

Fact 2.1.9 (Lemma 2.2 of Knight [23]). *Let $\overline{a} \in \mathfrak{A}$ and $\overline{b} \in \mathfrak{B}$. Suppose $p(\overline{a}) = q(\overline{b}) = \overline{k}$. If $\overline{a}_{\mathfrak{A}} \sim_n \overline{b}_{\mathfrak{B}}$, then $p \Vdash_{\mathfrak{A}} \varphi_e^{D^{(n)}}(k) = l$ iff $q \Vdash_{\mathfrak{B}} \varphi_e^{D^{(n)}}(k) = l$.*

Now we have the following analogue to Lemma 2.3 of [23].

Lemma 2.1.10. *Let $\overline{a} \in |\mathfrak{A}|$ and $\overline{b} \in |\mathfrak{B}|$. Suppose $p(\overline{a}) = q(\overline{b}) = \overline{k}$. If $\overline{a}_{\mathfrak{A}} \sim_{n+1} \overline{b}_{\mathfrak{B}}$, then $\Diamond_{\mathfrak{A}}^{p} W_e^{D^{(n)}} = S$ iff $\Diamond_{\mathfrak{B}}^{q} W_e^{D^{(n)}} = S$.*

Proof. Suppose that $\overline{a}_{\mathfrak{A}} \sim_{n+1} \overline{b}_{\mathfrak{B}}$, and let

$$P_{\mathfrak{A}} = \{k \mid (\exists p' \supseteq p)(\exists l) p' \Vdash_{\mathfrak{A}} \varphi_e^{D^{(n)}}(k) = l\};$$

$$P_{\mathfrak{B}} = \{k \mid (\exists q' \supseteq q)(\exists l) q' \Vdash_{\mathfrak{B}} \varphi_e^{D^{(n)}}(k) = l\}.$$

Since $p(\overline{a}) = q(\overline{b}) = \overline{k}$ and $\overline{a}_{\mathfrak{A}} \sim_{n+1} \overline{b}_{\mathfrak{B}}$, we have that for every $p' \supseteq p$, there is a $q' \supseteq q$ such $\mathrm{dom}(p')_{\mathfrak{A}} \sim_n \mathrm{dom}(q')_{\mathfrak{B}}$, and for every $q' \supseteq q$, there is a $p' \supseteq p$ such that $\mathrm{dom}(p')_{\mathfrak{A}} \sim_n \mathrm{dom}(q')_{\mathfrak{B}}$. Therefore, Fact 2.1.9 shows that for any k, $(\exists p' \supseteq p)(\exists l) p \Vdash_{\mathfrak{A}} \varphi_e^{D^{(n)}}(k) = l$ iff $(\exists q' \supseteq q)(\exists l) q' \Vdash_{\mathfrak{B}} \varphi_e^{D^{(n)}}(k) = l$. So $P_{\mathfrak{A}} = P_{\mathfrak{B}}$. $\square$

The following are the final two facts we need before proving Theorem 2.1.14. To state the first fact, we define the following terminology.

Definition 2.1.11. A linear order $\mathfrak{A}$ has the $\sim_n$*-property* if there is a computable linear order $\mathfrak{B}$ such that $\mathfrak{A} \sim_n \mathfrak{B}$.

Fact 2.1.12 (Lemma 3.4 of Knight [23])**.** *Every linear order has the* $\sim_2$*-property.*

Fact 2.1.13 (Lemma 3.2 of Knight [23])**.** *Let* $\overline{a} \in |\mathfrak{A}|$ *and* $\overline{b} \in |\mathfrak{B}|$*. Let* I_i *and* J_i *be the intervals in* $\mathfrak{A}$ *and* $\mathfrak{B}$*, respectively, such that* $\mathfrak{A} = I_0 + \{a_1\} + I_1 + \cdots + \{a_n\} + I_n$ *and* $\mathfrak{B} = J_0 + \{b_1\} + J_1 + \cdots + \{b_n\} + J_n$*. Then* $\overline{a}_{\mathfrak{A}} \sim_n \overline{b}_{\mathfrak{B}}$ *iff* $I_i \sim_n J_i$ *for all* $i = 0, \ldots, n$*.*

Theorem 2.1.14. *If* $\mathfrak{A}$ *is a linear order and* $S \subseteq \mathbb{N}$ *such that* $S \in \Sigma_1^{D(\mathfrak{B})'}$ *for all* $\mathfrak{B} \cong \mathfrak{A}$*, then* $S \in \Sigma_2^0$*.*

Proof. Let $\mathfrak{A}$ and S be as in the hypothesis. By Lemma 2.1.8, there is a $p \in F$ such that $\lozenge_{\mathfrak{A}}^p W_e^{D'} = S$. Let $\mathrm{dom}(p)$ consist of $a_0 < \cdots < a_{n-1}$. For $0 < i < n$, let I_i be the interval (a_{i-1}, a_i). Let I_0 be $(-\infty, a_0)$ if a_0 is not the first element of $\mathfrak{A}$, and $\emptyset$ otherwise; and let I_n be (a_n, ∞) if a_n is not the last element of $\mathfrak{A}$ and $\emptyset$ otherwise. There are computable orderings J_i such that $J_i \sim_2 I_i$, by Fact 2.1.12. Replacing each I_i by J_i produces a computable ordering $\mathfrak{B}$ such that $\mathrm{dom}(p)_{\mathfrak{B}} \sim_2 \mathrm{dom}(p)_{\mathfrak{A}}$. (This uses Fact 2.1.13.) Then $\lozenge_{\mathfrak{B}}^p W_e^{D'} = S$, and $S \in \Sigma_1^{D(\mathfrak{B})'} = \Sigma_2^0$. $\square$

2.2 Finite-component graphs

In order to separate the degree spectra of linear orders and other structures in the sense of the statement, "There is a linear order whose degree spectrum is not that of any __(name of other structure here)__," we can now apply Theorem 2.1.14 to different types of structures. Although the author now views finite-component graphs as merely an application of Theorem 2.1.14, the question of separating the degree spectra of linear orders and finite-component graphs was really one of the catalysts for this entire dissertation.

Now, using the definition of a countable finite-component graph from Section 1.2, note that the set

$$S_{\mathfrak{G}} = \{(C, n) \mid C \text{ is a component of } \mathfrak{G} \text{ occurring at least } n \text{ times}\}$$

is clearly an isomorphic invariant of $\mathfrak{G}$ (i.e., if $\mathfrak{B} \cong \mathfrak{G}$, then $S_{\mathfrak{B}} = S_{\mathfrak{G}}$). The following two properties of finite-component graphs will yield our desired application.

Lemma 2.2.1. *Let $\mathfrak{G}$ be a finite-component graph. Then $S_{\mathfrak{G}} \in \Sigma_1^{D(\mathfrak{G})'}$.*

Proof. Let $D(\mathfrak{G})'$ enumerate $S_{\mathfrak{G}}$: At every stage s, print out (C, n) if there is a connected subgraph C of $\mathfrak{G}_s$ (i.e., the graph determined by $D(\mathfrak{G})_s$) that occurs n times and that has stopped growing (a condition that $D(\mathfrak{G})'$ can correctly test) in each of those occurrences. □

Lemma 2.2.2. *If $\mathfrak{G}$ is a finite-component graph, $X \subseteq \mathbb{N}$, and $S_{\mathfrak{G}} \in \Sigma_1^X$, then there is a $\mathfrak{B} \cong \mathfrak{G}$ such that $D(\mathfrak{B}) \leq_T X$.*

Proof. Build $\mathfrak{B}$ by adding components as X enumerates them into $S_{\mathfrak{G}}$. □

Now we have the following corollary of Theorem 2.1.14. Note, we say that a degree $\mathbf{d}$ *computes* a structure $\mathfrak{A}$ if $\deg(\mathfrak{A}) \leq \mathbf{d}$.

Theorem 2.2.3. *If $\mathfrak{A}$ is a linear order and $\mathfrak{G}$ is a finite-component graph such that $\mathrm{DgSp}(\mathfrak{A}) \subseteq \mathrm{DgSp}(\mathfrak{G})$, then $\mathbf{0}'$ computes a copy of $\mathfrak{G}$ (and so $\{\mathbf{d} \mid \mathbf{d} \geq \mathbf{0}'\} \subseteq \mathrm{DgSp}(\mathfrak{G})$).*

Proof. Let $\mathfrak{A}$ and $\mathfrak{G}$ be as in the hypothesis. Fix a copy $\mathfrak{B}$ of $\mathfrak{A}$. Let $\hat{\mathfrak{G}} \cong \mathfrak{G}$ such that $D(\hat{\mathfrak{G}}) \leq_T D(\mathfrak{B})$. Then by Lemma 2.2.1, $S_{\mathfrak{G}} = S_{\hat{\mathfrak{G}}} \in \Sigma_1^{D(\hat{\mathfrak{G}})'} \subseteq \Sigma_1^{D(\mathfrak{B})'}$. So by Theorem 2.1.14, $S_{\mathfrak{G}} \in \Sigma_1^{\emptyset'}$. Thus Lemma 2.2.2 says there is a $\tilde{\mathfrak{G}} \cong \mathfrak{G}$ such that $D(\tilde{\mathfrak{G}}) \leq_T \emptyset'$, so $\mathbf{0}'$ computes a copy of $\mathfrak{G}$. Since degree spectra are closed upward, $\{\mathbf{d} \mid \mathbf{d} \geq \mathbf{0}'\} \subseteq \mathrm{DgSp}(\mathfrak{G})$. □

Finally, the following lemma on linear orders gives us the desired degree spectrum separation from finite-component graphs.

Lemma 2.2.4. *For any degree $\mathbf{d}$, there is a linear order $\mathfrak{L}$ such that $\mathbf{d}$ cannot compute a copy of $\mathfrak{L}$.*

Proof. Fix $\mathbf{d}$, and for some $D \in \mathbf{d}$, let $X = D^{(4)}$ such that $0 \notin X$, without loss of generality. Write $X = \{x_0 < x_1 < x_2 < \cdots\}$. Define $\zeta(X)$ to be the linear order:

$$\mathbb{Z} + x_0 + \mathbb{Z} + x_1 + \mathbb{Z} + x_2 + \cdots$$

For a linear order $\mathfrak{L}$, let $\mathrm{Succ}_{\mathfrak{L}}$ and $\mathrm{Bl}_{\mathfrak{L}}$ denote the successivity and block relation for $\mathfrak{L}$ (where a *successivity* is a point paired with its successor and a *block* is a string of successive points where the first point has no predecessor and the last point has no successor). Then for all $a_1, \ldots, a_n \in |\mathfrak{L}|$,

$$\mathrm{Succ}_{\mathfrak{L}}(a_1, a_2) \iff a_1 <_{\mathfrak{L}} a_2 \wedge (\neg\exists c)[a_1 <_{\mathfrak{L}} c <_{\mathfrak{L}} a_2];$$

$$\mathrm{Bl}_{\mathfrak{L}}(a_1, \ldots, a_n) \iff (\forall b)\neg \mathrm{Succ}_{\mathfrak{L}}(b, a_1) \wedge (\forall b)\neg \mathrm{Succ}_{\mathfrak{L}}(a_n, b) \wedge \bigwedge_{i<j} \mathrm{Succ}_{\mathfrak{L}}(a_i, a_j).$$

Let $\hat{\mathfrak{L}} \cong \zeta(X)$. Then, for all $n \geq 1$,

$$\begin{aligned} n \in X &\iff (\exists a_1, \ldots, a_n)\, \mathrm{Bl}_{\zeta(X)}(a_1, \ldots, a_n) \\ &\iff (\exists a_1, \ldots, a_n)\, \mathrm{Bl}_{\hat{\mathfrak{L}}}(a_1, \ldots, a_n). \end{aligned}$$

Then $X \in \Sigma_3^{D(\hat{\mathfrak{L}})}$, so $X \leq_T D(\hat{\mathfrak{L}})'''$. Clearly $\deg(\hat{\mathfrak{L}}) \not\leq \mathbf{d}$. Therefore, $\mathbf{d}$ cannot compute a copy of $\mathfrak{L} = \zeta(X)$. □

Theorem 2.2.5. *There is a linear order* $\mathfrak{L}$ *such that* $\mathrm{DgSp}(\mathfrak{L}) \neq \mathrm{DgSp}(\mathfrak{G})$ *for any finite-component graph* $\mathfrak{G}$*.*

Proof. From Lemma 2.2.4, let $\mathfrak{L}$ be a linear order such that $\mathbf{0}'$ cannot compute a copy of $\mathfrak{L}$. For a contradiction, assume that $\mathrm{DgSp}(\mathfrak{L}) = \mathrm{DgSp}(\mathfrak{G})$ for some finite-component graph $\mathfrak{G}$. Then by Theorem 2.2.3, $\mathbf{d} \in \mathrm{DgSp}(\mathfrak{G})$ for some $\mathbf{d} \leq \mathbf{0}'$. So $\mathbf{d} \in \mathrm{DgSp}(\mathfrak{L})$, and $\mathbf{0}'$ computes a copy of $\mathfrak{L}$. □

2.3 Equivalence structures

For an equivalence structure $\mathfrak{E}$, let

$$S_{\mathfrak{E}} = \{(n, m) \mid \mathfrak{E} \text{ has at least } m \text{ many classes of size exactly } n\}.$$

Again, if $\mathfrak{E} \cong \mathfrak{F}$, then $S_{\mathfrak{E}} = S_{\mathfrak{F}}$. Also, the isomorphism type of $\mathfrak{E}$ is determined by $S_{\mathfrak{E}}$ together with the number of infinite equivalence classes. To obtain the same result as we did for finite-component graphs now for equivalence structures, we show the following two analogous lemmas.

Lemma 2.3.1. *If $\mathfrak{E}$ is an equivalence structure, then $S_{\mathfrak{E}} \in \Sigma_1^{D(\mathfrak{E})'}$.*

Proof. Let $D(\mathfrak{E})'$ enumerate $D(\mathfrak{E})$. At stage s, print out (n, m) if there are m many equivalence classes of size n in $\mathfrak{E}_s$ (i.e., the equivalence structure determined by $D(\mathfrak{E})_s$), each of which has stopped growing (a condition that $D(\mathfrak{E})'$ can correctly test). Notice that nothing will be printed in the case of an infinite equivalence class of $\mathfrak{E}$, since they never stop growing. □

Lemma 2.3.2. *If $\mathfrak{E}$ is an equivalence structure, $X \subseteq \mathbb{N}$, and $S_{\mathfrak{E}} \in \Sigma_1^X$, then there is an $\mathfrak{F} \cong \mathfrak{E}$ such that $D(\mathfrak{F}) \leq_T X$.*

Proof. Build $\mathfrak{F}$ by adding equivalence classes as X enumerates them into $S_{\mathfrak{E}}$. Also, nonuniformly add infinite classes to $\mathfrak{F}$ so that $\mathfrak{E}$ and $\mathfrak{F}$ have the same number of infinite equivalence classes. □

Then by the exact same reasoning as in the proofs of Theorems 2.2.3 and 2.2.5, we have the following two theorems.

Theorem 2.3.3. *If* $\mathfrak{A}$ *is a linear order and* $\mathfrak{E}$ *is an equivalence structure such that* $\mathrm{DgSp}(\mathfrak{A}) \subseteq \mathrm{DgSp}(\mathfrak{E})$*, then* $\mathbf{0}'$ *computes a copy of* $\mathfrak{E}$ *(and so* $\{\mathbf{d} \mid \mathbf{d} \geq \mathbf{0}'\} \subseteq \mathrm{DgSp}(\mathfrak{E})$*).*

Proof. Let $\mathfrak{A}$ and $\mathfrak{E}$ be as in the hypothesis. Fix a copy $\mathfrak{B}$ of $\mathfrak{A}$. Let $\hat{\mathfrak{E}} \cong \mathfrak{E}$ such that $D(\hat{\mathfrak{E}}) \leq_T D(\mathfrak{B})$. Then by Lemma 2.3.1, $S_{\mathfrak{E}} = S_{\hat{\mathfrak{E}}} \in \Sigma_1^{D(\hat{\mathfrak{E}})'} \subseteq \Sigma_1^{D(\mathfrak{B})'}$. So by Theorem 2.1.14, $S_{\mathfrak{E}} \in \Sigma_1^{\emptyset'}$. Thus Lemma 2.3.2 says there is an $\tilde{\mathfrak{E}} \cong \mathfrak{E}$ such that $D(\tilde{\mathfrak{E}}) \leq_T \emptyset'$, so $\mathbf{0}'$ computes a copy of $\mathfrak{E}$. □

Theorem 2.3.4. *There is a linear order* $\mathfrak{L}$ *such that* $\mathrm{DgSp}(\mathfrak{L}) \neq \mathrm{DgSp}(\mathfrak{E})$ *for any equivalence structure* $\mathfrak{E}$*.*

Proof. From Lemma 2.2.4, let $\mathfrak{L}$ be a linear order such that $\mathbf{0}'$ cannot compute a copy of $\mathfrak{L}$. For a contradiction, assume that $\mathrm{DgSp}(\mathfrak{L}) = \mathrm{DgSp}(\mathfrak{E})$ for some equivalence structure $\mathfrak{E}$. Then by Theorem 2.3.3, $\mathbf{d} \in \mathrm{DgSp}(\mathfrak{E})$ for some $\mathbf{d} \leq \mathbf{0}'$. So $\mathbf{d} \in \mathrm{DgSp}(\mathfrak{L})$, and $\mathbf{0}'$ computes a copy of $\mathfrak{L}$. □

2.4 Rank-1 torsion-free abelian groups

We can also obtain the desired separation of degree spectra in the case of linear orders vs. rank-1 torsion-free abelian groups. First, note that we have the following analogue of Theorem 2.1.14.

Theorem 2.4.1. *If $\mathfrak{A}$ is a linear order and $S \subseteq \mathbb{N}$ such that $S \in \Sigma_1^{D(\mathfrak{B})}$ for all $\mathfrak{B} \cong \mathfrak{A}$, then $S \in \Sigma_1^0$.*

The proof is the same as that of Theorem 2.1.14 presented in Section 2.1 without the jump. Recall the definition of a *rank-1 torsion-free abelian group* from Section 1.2. We now define the *standard type* of any subgroup of $\mathbb{Q}$ as in Coles, Downey, and Slaman [7]. This will play the role of the set S in the lemmas to come.

Definition 2.4.2. Let $p_1 < p_2 < \ldots$ be the primes, and let $\mathfrak{G}$ be a subgroup of $\mathbb{Q}$.

(1) For a prime p, the *p-height* $h_p(a)$ of an $a \in |\mathfrak{G}|$, $a \neq 0_{\mathfrak{G}}$, is

$$h_p(a) = \begin{cases} k, & \text{if } k \text{ is greatest such that } p^k | a \text{ in } \mathfrak{G}, \\ \infty, & \text{if } p^k | a \text{ for all } k. \end{cases}$$

(2) The *characteristic* of a is the sequence

$$\chi(a) = (h_{p_1}(a), h_{p_2}(a), h_{p_3}(a), \ldots).$$

(3) The *standard type* $S(\mathfrak{G})$ of $\mathfrak{G}$ relative to a fixed $a \in |\mathfrak{G}|$, $a \neq 0_{\mathfrak{G}}$, is

$$S(\mathfrak{G}) = \{(i, j) \mid (\exists g)(p_i^j g = a)\}.$$

Now we have the two lemmas for rank-1 torsion-free abelian groups which are the analogues of Lemmas 2.2.1 and 2.2.2 for finite-component graphs and Lemmas 2.3.1 and 2.3.2 for equivalence structures.

Lemma 2.4.3. *If $\mathfrak{G}$ is a rank-1 torsion-free abelian group, then $S(\mathfrak{G}) \in \Sigma_1^{D(\mathfrak{B})}$ for all $\mathfrak{B} \cong \mathfrak{G}$.*

Proof. Nonuniformly fix a $b \in |\mathfrak{B}|$ that corresponds to $a \in |\mathfrak{G}|$. Then

$$S(\mathfrak{G}) = \{(i,j) \mid (\exists g \in |\mathfrak{B}|)(p_i^j g = b)\}$$

is clearly $\Sigma_1^{D(\mathfrak{B})}$. □

See the proof of Theorem 17 in [7] for the proof of this second lemma.

Lemma 2.4.4. *If $\mathfrak{G}$ is a rank-1 torsion-free abelian group, $X \subseteq \mathbb{N}$, and $S(\mathfrak{G}) \in \Sigma_1^X$, then there is a $\mathfrak{B} \cong \mathfrak{G}$ such that $D(\mathfrak{B}) \leq_T X$.*

By reasoning familiar now from the cases of finite-component graphs and equivalence structures, we have the following two theorems.

Theorem 2.4.5. *If $\mathfrak{A}$ is a linear order and $\mathfrak{G}$ is a rank-1 torsion-free abelian group such that* $\mathrm{DgSp}(\mathfrak{A}) \subseteq \mathrm{DgSp}(\mathfrak{G})$*, then* $\mathbf{0}$ *computes a copy of $\mathfrak{G}$ (and so* $\mathbf{D} = \mathrm{DgSp}(\mathfrak{G})$*).*

Proof. Let $\mathfrak{A}$ and $\mathfrak{G}$ be as in the hypothesis. Fix a copy $\mathfrak{B}$ of $\mathfrak{A}$. Let $\hat{\mathfrak{G}} \cong \mathfrak{G}$ such that $D(\hat{\mathfrak{G}}) \leq_T D(\mathfrak{B})$. Then by Lemma 2.4.3, $S_{\mathfrak{G}} = S_{\hat{\mathfrak{G}}} \in \Sigma_1^{D(\hat{\mathfrak{G}})} \subseteq \Sigma_1^{D(\mathfrak{B})}$. So by Theorem 2.4.1, $S_{\mathfrak{G}} \in \Sigma_1^0$. Thus Lemma 2.4.4 says there is a $\tilde{\mathfrak{G}} \cong \mathfrak{G}$ such that $D(\tilde{\mathfrak{G}}) \leq_T \emptyset$, so $\mathbf{0}$ computes a copy of $\mathfrak{G}$. □

Theorem 2.4.6. *There is a linear order $\mathfrak{L}$ such that* $\mathrm{DgSp}(\mathfrak{L}) \neq \mathrm{DgSp}(\mathfrak{G})$ *for any rank-1 torsion-free abelian group $\mathfrak{G}$. In fact, the* only *spectrum that can be* *the spectrum of* both *a linear order and a rank-1 torsion-free abelian group is the trivial spectrum (i.e., the spectrum of a computable structure).*

Proof. From Lemma 2.2.4, let $\mathfrak{L}$ be a linear order such that $\mathbf{0}$ cannot compute a copy of $\mathfrak{L}$. For a contradiction, assume that $\mathrm{DgSp}(\mathfrak{L}) = \mathrm{DgSp}(\mathfrak{G})$ for some rank-1 torsion-free abelian group $\mathfrak{G}$. Then by Theorem 2.4.5, $\mathbf{0} \in \mathrm{DgSp}(\mathfrak{G})$. So $\mathbf{0} \in \mathrm{DgSp}(\mathfrak{L})$, and $\mathbf{0}$ computes a copy of $\mathfrak{L}$. $\square$

Chapter 3

Families of sets

In this chapter, we perform the same analysis on degree spectra of structures as we did in Chapter 2, only now we work with enumerations of countable *families of sets* instead of enumerations of countable *sets*. This will add to our list of examples—in the sense of Chapter 2, we can separate the degree spectrum of linear orders from that of finite-component graphs, equivalence structures, rank-1 torsion-free abelian groups, and now *daisy graphs*, as presented in Section 3.2.

To see exactly how the methods of this chapter will extend those of Chapter 2, notice later that within the proof of our main separation of degree spectra result, Theorem 3.2.9, we prove the family-of-sets analogue to Theorem 2.1.14. That is, we prove that if S is a countable family of sets and if $\mathfrak{L}$ is a linear order such that $\mathfrak{L}$ has the $\sim_3$-property and such that for every $\mathfrak{B} \cong \mathfrak{L}$, S has an enumeration $\nu_{\mathfrak{B}} \in \Sigma_1^{D(\mathfrak{B})}$, then S has an enumeration $\nu \in \Sigma_1^{\emptyset''}$.

3.1 Extending to enumerations of families of sets

Definition 3.1.1. Let $S \subseteq \mathcal{P}(\mathbb{N})$ be a countable family of subsets of $\mathbb{N}$. A binary relation ν is called an *enumeration* of S if $S = \{\nu(i) \mid i \in \mathbb{N}\}$, where $\nu(i) = \{x \mid (i,x) \in \nu\}$.

We use the same forcing setup as in Section 2.1 (where we fix a structure $\mathfrak{A}$ and build a generic copy $\mathfrak{B}$ via a chain of forcing conditions), and we use the same forcing statements as in Definitions 2.1.2 and 2.1.4. Let $W_e^{D(\mathfrak{B})^{(m)}}[n] = \{x \mid (n,x) \in W_e^{D(\mathfrak{B})^{(m)}}\}$ denote the *n-th column of* $W_e^{D(\mathfrak{B})^{(m)}}$. Add the following informal forcing statements to our list in Definitions 2.1.2 and 2.1.4. Again, we use $\Diamond_{\mathfrak{A}}^p$ in place of $p \Vdash_{\mathfrak{A}}$ when defining a quasi forcing statement.

Definition 3.1.2. Let $p \Vdash_{\mathfrak{A}} \psi$ and $\Diamond_{\mathfrak{A}}^p \psi$ be defined as follows.

6. ψ is $W_e^{D^{(m)}}[n] = X$, for some $X \subseteq \mathbb{N}$ and some n, m:

   $\Diamond_{\mathfrak{A}}^p \psi \iff X = \{k \mid (\exists r \supseteq p)(\exists l) r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l\}$.

   $p \Vdash_{\mathfrak{A}} \psi \iff (\forall q \supseteq p) \Diamond_{\mathfrak{A}}^q W_e^{D^{(m)}}[n] = X$.

   $p \Vdash_{\mathfrak{A}} \neg\psi \iff$ there is no $q \supseteq p$ such that $q \Vdash_{\mathfrak{A}} \psi$.

7. ψ is $W_e^{D^{(m)}}$ enumerates S, for some countable $S \subseteq \mathcal{P}(\mathbb{N})$ and some m:

   $\Diamond_{\mathfrak{A}}^p \psi \iff S = \{X \mid (\exists n)(\exists q \supseteq p) q \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] = X\}$.

The following three lemmas, where $\mathfrak{A}$ and $\mathfrak{B}$ are linear orders and S is a countable subset of $\mathcal{P}(\mathbb{N})$, will yield our degree spectra separation result for daisy graphs, Theorem 3.2.9, along with our result linking degree spectra to families of sets in general, Theorem 3.2.12.

Lemma 3.1.3. *If, for all $\mathfrak{B} \cong \mathfrak{A}$, S has an enumeration c.e. in $D(\mathfrak{B})^{(m)}$, then there are p and e such that $\Diamond_{\mathfrak{A}}^{p} W_e^{D^{(m)}}$ enumerates S.*

Proof. Assume for every p and e, the condition that $\Diamond_{\mathfrak{A}}^{p} W_e^{D^{(m)}}$ enumerates S fails. It suffices to show that there is an m-complete sequence $(p_i)_{i \in \mathbb{N}}$ that determines a generic copy $\mathfrak{B}$ of $\mathfrak{A}$ such that S has no enumeration c.e. in $D(\mathfrak{B})^{(m)}$. By our assumption, the following holds for every p and e:

$$(\exists X \in S)(\forall n)(\forall q \supseteq p) q \not\Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] = X \tag{3.1.1}$$

or

$$(\exists X \notin S)(\exists n)(\exists q \supseteq p) q \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] = X \tag{3.1.2}$$

Now we can state exactly what requirements need to be met by the sequence of forcing conditions p_e, noting that interleaved with what we are actually doing in the construction, there are other stages making the sequence m-complete (just as in the proof of Lemma 2.1.8). For each e, we have a master requirement that needs to make $W_e^{D(\mathfrak{B})^{(m)}}$ not an enumeration of the family S. This master requirement will be met in one of two ways. Either it will make sure that there is an $X \notin S$, a condition p_i, and a number $n \in \mathbb{N}$ such that

$$p_i \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] = X,$$

or it will make sure that there is an $X \in S$ and an infinite sequence of conditions p_{i_n} for $n \in \mathbb{N}$ such that for each n,

$$p_{i_n} \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] \neq X$$

(so that X is not *any* column of $W_e^{D(\mathfrak{B})^{(m)}}$).

To build $(p_i)_{i \in \mathbb{N}}$, let $p_0 = \emptyset$. Suppose that we have defined p_i and that the next master requirement to meet is for the index e. If (3.1.2) holds, then let p_{i+1} be a $q \supseteq p_i$ such that $q \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] = X$ for an $n \in \mathbb{N}$ and $X \notin S$ guaranteed by (3.1.2). Notice that if $(p_i)_{i \in \mathbb{N}}$ is m-complete, then each statement of the form $\varphi_e^{D^{(m)}}(n,k)\downarrow$ is eventually forced or its negation is forced. If $p_{i+1} \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] = X$, then there cannot be an extension forcing $\varphi_e^{D^{(m)}}(n,k)\uparrow$ if $k \in X$. So we must have $\varphi_e^{D^{(m)}}(n,k)\downarrow$ eventually forced. This action satisfies the master requirement.

Otherwise, suppose that (3.1.1) holds. Fix an $X \in S$ such that for every n and every $q \supseteq p_i$,

$$(\exists q' \supseteq q) X \neq \{k \mid (\exists r \supseteq q')(\exists l) r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l\}. \tag{3.1.3}$$

Define infinitely many subrequirements corresponding to the infinite sequence p_{i_n}. Let the master requirement dovetail the action of these requirements with the action of future master requirements.

To meet the subrequirement for n given a current forcing condition p_i, consider a $q' \supseteq p_i$ such that $X \neq \{k \mid (\exists r \supseteq q')(\exists l) r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l\}$ as guaranteed by (3.1.3). For such a q', we have either

$$(\exists k \in X)(\forall r \supseteq q')(\forall l) r \nVdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l \tag{3.1.4}$$

or

$$(\exists k \notin X)(\exists r \supseteq q')(\exists l) r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l. \tag{3.1.5}$$

If (3.1.5) holds, then let p_{i+1} be an $r \supseteq q' \supseteq p_i$ such that $r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l$ for an $l \in \mathbb{N}$ and $k \notin X$ as guaranteed by (3.1.5). If (3.1.4) holds instead, let $p_{i+1} = q'$. Fix $k \in X$ such that

$$(\forall r \supseteq p_{i+1})(\forall l) r \nVdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l,$$

and we have $p_{i+1} \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k)\uparrow$, so because $k \in X$, $p_{i+1} \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] \neq X$.

Thus the dovetailed subrequirements get met one by one, and at the end of the construction, the master requirement is met. So $(p_i)_{i\in\mathbb{N}}$ determines a generic $\mathfrak{B} \cong \mathfrak{A}$ such that S has no enumeration c.e. in $D(\mathfrak{B})^{(m)}$ as desired. □

Lemma 3.1.4. *If $\Diamond_{\mathfrak{B}}^{p} W_e^{D^{(m)}}$ enumerates S, then S has an enumeration c.e. in $D(\mathfrak{B})^{(m+2)}$.*

Proof. Suppose that $\Diamond_{\mathfrak{B}}^{q} W_e^{D^{(m)}}$ enumerates S. Then

$$S = \{X \mid (\exists n)(\exists q \supseteq p) q \Vdash_{\mathfrak{B}} W_e^{D^{(m)}}[n] = X\}.$$

Now for any set X, $q \Vdash_{\mathfrak{B}} W_e^{D^{(m)}}[n] = X$ iff for each $k \in X$,

$$(\forall q' \supseteq q)(\exists r \supseteq q')(\exists l) r \Vdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l \tag{3.1.6}$$

and for each $k \notin X$,

$$(\forall r \supseteq q)(\forall l) r \nVdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l. \tag{3.1.7}$$

Thus, we can capture the notion of a condition q forcing $W_e^{D^{(m)}}[n]$ being equal to some set. Indeed, we say

$$q \Vdash_{\mathfrak{B}} W_e^{D^{(m)}}[n] \text{ is determined} :\Longleftrightarrow (\forall k)[(3.1.6) \text{ holds} \vee (3.1.7) \text{ holds}]. \tag{3.1.8}$$

So to enumerate S, we need to find the conditions $q \supseteq p$ such that $q \Vdash_{\mathfrak{B}} W_e^{D^{(m)}}[n]$ is determined and then enumerate the sets

$$\{k \mid (\exists r \supseteq q)(\exists l) r \Vdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l\}$$

for such conditions q. Since (by Fact 2.1.5 and the same reasoning as in Lemma 2.1.7) the relation $r \Vdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l$ is c.e. in $D(\mathfrak{B})^{(m)}$, $D(\mathfrak{B})^{(m+2)}$ can compute whether $q \Vdash_{\mathfrak{B}} W_e^{D^{(m)}}[n]$ is determined (by the complexity of the definition in (3.1.8)). Therefore, $D(\mathfrak{B})^{(m+2)}$ can enumerate the sets determined by an extension of p. $\square$

Lemma 3.1.5. *Let $\overline{a} \in |\mathfrak{A}|$ and $\overline{b} \in |\mathfrak{B}|$. Suppose $p(\overline{a}) = q(\overline{a}) = \overline{k}$. If $\Diamond_{\mathfrak{A}}^{p} W_e^{D^{(m)}}$ enumerates S and $\overline{a}_{\mathfrak{A}} \sim_{m+3} \overline{b}_{\mathfrak{B}}$, then $\Diamond_{\mathfrak{B}}^{q} W_e^{D^{(m)}}$ enumerates S.*

Proof. Suppose that $\Diamond_{\mathfrak{A}}^{p} W_e^{D^{(m)}}$ enumerates S and $\overline{a}_{\mathfrak{A}} \sim_{m+3} \overline{b}_{\mathfrak{B}}$. Define

$$P_{\mathfrak{A}} = \{X \mid (\exists n)(\exists r \supseteq p) r \Vdash_{\mathfrak{A}} W_e^{D^{(m)}}[n] = X\};$$

$$P_{\mathfrak{B}} = \{X \mid (\exists n)(\exists r \supseteq q) r \Vdash_{\mathfrak{B}} W_e^{D^{(m)}}[n] = X\}.$$

Then $S = P_{\mathfrak{A}}$ and we wish to prove $S = P_{\mathfrak{B}}$, so it suffices to show that $P_{\mathfrak{A}} = P_{\mathfrak{B}}$. Now, $X \in P_{\mathfrak{A}}$ iff

$$(\exists n)(\exists p_1 \supseteq p)(\forall p_2 \supseteq p_1)X = \{k \mid (\exists r \supseteq p_2)(\exists l) r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l\}, \quad (3.1.9)$$

and $X \in P_{\mathfrak{B}}$ iff

$$(\exists n)(\exists q_1 \supseteq q)(\forall q_2 \supseteq q_1)X = \{k \mid (\exists r' \supseteq q_2)(\exists l) r' \Vdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l\}.$$

Let $X \in P_{\mathfrak{A}}$, and let n and $p_1 \supseteq p$ be as in (3.1.9). Then there is a $q_1 \supseteq q$ such that $\mathrm{dom}(p_1)_{\mathfrak{A}} \sim_{m+2} \mathrm{dom}(q_1)$. Let $q_2 \supseteq q_1$. Then there is a $p_2 \supseteq p_1$ such that $\mathrm{dom}(p_2)_{\mathfrak{A}} \sim_{m+1} \mathrm{dom}(q_2)$. Therefore,

$$X = \{k \mid (\exists r \supseteq p_2)(\exists l) r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l\}$$

by (3.1.9).

We claim that

$$X = \{k \mid (\exists r' \supseteq q_2)(\exists l) r' \Vdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l\}.$$

On one hand, let $k \in X$. So take $r \supseteq p_2$ such that $r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l$ for some l. Then there is an $r' \supseteq q_2$ such that $\mathrm{dom}(r)_{\mathfrak{A}} \sim_m \mathrm{dom}(r')_{\mathfrak{B}}$. By Fact 2.1.9, $r' \Vdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l$. On the other hand, let k and $r' \supseteq q_2$ be such that $r' \Vdash_{\mathfrak{B}} \varphi_e^{D^{(m)}}(n,k) = l$ for some l. Then there is an $r \supseteq p_2$ such that $\mathrm{dom}(r)_{\mathfrak{A}} \sim_m \mathrm{dom}(r')_{\mathfrak{B}}$; hence, $r \Vdash_{\mathfrak{A}} \varphi_e^{D^{(m)}}(n,k) = l$. So $k \in X$.

Therefore, since $q_2 \supseteq q_1$ was arbitrary, $X \in P_{\mathfrak{B}}$. Notice that the same argument shows that $X \in P_{\mathfrak{B}} \implies X \in P_{\mathfrak{A}}$. Hence $P_{\mathfrak{A}} = P_{\mathfrak{B}}$. $\square$

3.2 Application to Daisy Graphs

We can now show the separation of degree spectra for daisy graphs. We define a *daisy graph* of a countable family S of sets in a similar way as in Goncharov, et al. [17]. We are interested in two kinds of daisy graphs, those that repeat each set in S infinitely many times and those that mention each set in S only once. We therefore consider not only enumerations of S, but *1-1 enumerations* of S, which are also defined below.

Definition 3.2.1. Let $S \subseteq \mathcal{P}(\mathbb{N})$ be a countable family of sets.

- For each $X \in S$, define $\mathfrak{G}(X)$ to be the directed graph consisting of one *root* node r at the center, with $r \to r$, and for each $x \in X$, a *petal* of the form

$$r \to x_0 \to \cdots \to x_n \to r.$$

  The petals are disjoint except for the root node, which is common to all.

- Let the *repeat daisy graph* $\mathfrak{G}_\infty(S)$ of S be the union of the disjoint family of graphs $\mathfrak{G}(X)$ for $X \in S$, having infinitely many $\mathfrak{G}(X)$ for each X.

- Let the *1-1 daisy graph* $\mathfrak{G}(S)$ of S be the union of the disjoint family of graphs $\mathfrak{G}(X)$ for $X \in S$, having one $\mathfrak{G}(X)$ for each X.

- We say that $\mathfrak{G}$ is a *daisy graph* of S if $\mathfrak{G}$ is either $\mathfrak{G}_\infty(S)$ or $\mathfrak{G}(S)$.

- μ is a *1-1 enumeration* of S if μ is an enumeration of S and for all $i \neq j$, $\mu(i) \neq \mu(j)$.

As a brief remark on the computational complexity of daisy graphs, before investigating their degree spectra, there are daisy graphs with all possible *computable dimension*. The *computable dimension* of a computably presentable structure $\mathfrak{A}$ is the number of computable presentations of $\mathfrak{A}$ up to computable isomorphism. Indeed, Theorem 4.3 of [17] (due to Goncharov) states that for each n, there is a daisy graph of computable dimension n. Hence daisy graphs can exhibit complicated behavior, from a computable model theoretic standpoint, so it is worth noting that they are non-universal.

At the beginning of the chapter we mentioned that in the proof our main result for repeat daisy graphs, Theorem 3.2.9, we will show that if for every copy $\mathfrak{B}$ of a linear order that has the $\sim_3$-property, S has an enumeration $\nu_{\mathfrak{B}} \in \Sigma_1^{D(\mathfrak{B})}$, then S has an enumeration $\nu \in \Sigma_1^{\emptyset''}$. So to obtain a version of Theorem 3.2.9 for *1-1* daisy graphs, we will use this result and then finish the proof by employing the following lemma.

Lemma 3.2.2. *Let $S \subseteq \mathcal{P}(\mathbb{N})$ be a countable family of sets, and let ν be an enumeration of S. If $\nu \in \Sigma_1^{\emptyset^{(m)}}$, then there is a 1-1 enumeration μ of S such that $\mu \in \Sigma_1^{\emptyset^{(m+2)}}$. (In fact, μ is computable in $\emptyset^{(m+2)}$.)*

Proof. Suppose that $\nu \in \Sigma_1^{\emptyset^{(m)}}$. For $i, j \in \mathbb{N}$, we have $\nu(i) = \nu(j)$ iff

$$(\forall x)(\nu(i,x) \iff \nu(j,x)). \tag{3.2.1}$$

Since (3.2.1) is $\Pi_2^{\emptyset^{(m)}}$, $\emptyset^{(m+2)}$ can compute whether $\nu(i) = \nu(j)$. Define the 1-1 enumeration μ by $\mu(0) = \nu(0)$ and $\mu(i+1) = \nu(j)$, where j is least such that $\mu(0) \neq \nu(j), \ldots, \mu(i) \neq \nu(j)$. □

To obtain the full application to daisy graphs, we will use the following facts about linear order multiplication and the $\sim_n$ relation. For linear orders $\mathfrak{A}$ and $\mathfrak{B}$, the notation $\mathfrak{A} \cdot \mathfrak{B}$ denotes the replacement of each point of $\mathfrak{B}$ by a copy of $\mathfrak{A}$. Lemma 3.2.3 follows from an easy induction on n; Lemma 3.2.4 follows from Lemma 3.2.3, which we show; and Lemma 3.2.5 follows quickly from the definition of the $\sim_n$-relation.

Lemma 3.2.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be linear orderings. Suppose that $\overline{a} \in |\mathbb{Z} \cdot \mathfrak{A}|$, $\overline{b} \in |\mathbb{Z} \cdot \mathfrak{B}|$, and $|[a_i, a_j]| = k$ iff $|[b_i, b_j]| = k$. Then*

$$(\overline{a}/\approx)_{\mathfrak{A}} \sim_n (\overline{b}/\approx)_{\hat{\mathfrak{B}}} \implies \overline{a}_{\mathbb{Z}\cdot\mathfrak{A}} \sim_{n+1} \overline{b}_{\mathbb{Z}\cdot\mathfrak{B}},$$

where we say $c_i \approx c_j$ if $[c_i, c_j]$ is finite.

Lemma 3.2.4. *Let $n \in \mathbb{N}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are linear orders such that $\mathfrak{A} \sim_n \mathfrak{B}$, then for any $k \in \mathbb{N}$, $\mathbb{Z}^k \cdot \mathfrak{A} \sim_{n+k} \mathbb{Z}^k \cdot \mathfrak{B}$.*

Proof. Notice that by induction, it suffices to show $\mathfrak{A} \sim_n \mathfrak{B} \implies \mathbb{Z}\cdot\mathfrak{A} \sim_{n+1} \mathbb{Z}\cdot\mathfrak{B}$. Let $\overline{a_0} \in |\mathbb{Z} \cdot \mathfrak{A}|$. Pick $\overline{b_0} \in |\mathbb{Z} \cdot \mathfrak{B}|$ such that $(\overline{a_0}/\approx)_{\mathfrak{A}} \sim_n (\overline{b_0}/\approx)_{\mathfrak{B}}$ and such that if $a_i < a_j$ in some $\mathbb{Z}$ copy, then $b_i < b_j$ in some $\mathbb{Z}$ copy and $|[a_i, a_j]| = |[b_i, b_j]|$. Then by Lemma 3.2.3, $\overline{a}_{\mathbb{Z}\cdot\mathfrak{A}} \sim_{n+1} \overline{b}_{\mathbb{Z}\cdot\mathfrak{B}}$. □

Lemma 3.2.5. *Let $n \in \mathbb{N}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are linear orders such that $\mathfrak{A} \sim_n \mathfrak{B}$, then for any linear orders α and β, $\alpha + \mathfrak{A} + \beta \sim_n \alpha + \mathfrak{B} + \beta$.*

Now, keeping in mind how we obtained the results for finite-component graphs and our other examples, we have two immediate lemmas for daisy graphs that lead us to their separation from linear orders with respect to degree spectra. Notice the third lemma stated is a 1-1 version of Lemma 3.2.7 for the application to 1-1 daisy graphs.

Lemma 3.2.6. *If $\mathfrak{G}$ is a daisy graph of a countable family S of sets, then S has an enumeration in $\Sigma_1^{D(\mathfrak{G})}$.*

Proof. As we see a petal of size $x+2$ appear off of the i^{th} root node of $\mathfrak{G}$, put x into the i^{th} set of the enumeration. Eventually we will list an enumeration of S. □

Lemma 3.2.7. *Let $\mathfrak{G}$ be a repeat daisy graph of a countable family S of sets, and let $X \subseteq \mathbb{N}$. If S has an enumeration in Σ_1^X, then there is a $\hat{\mathfrak{G}} \cong \mathfrak{G}$ such that $D(\hat{\mathfrak{G}}) \leq_T X$.*

Proof. The copy $\hat{\mathfrak{G}}$ of $\mathfrak{G}$ is the graph built by letting X print out an enumeration ν of S and putting in the corresponding root nodes and petals as we see new sets and elements get listed, making sure to repeat each component infinitely many times to get $\mathfrak{G}_\infty(S)$. For X to correctly determine whether two vertices v_1, v_2 are connected by an edge in $\hat{\mathfrak{G}}$, print enough of ν to find the petal(s) containing v_1, v_2. □

Lemma 3.2.8. *Let $\mathfrak{G}$ be a 1-1 daisy graph of a countable family S of sets, and let $X \subseteq \mathbb{N}$. If S has a 1-1 enumeration in Σ_1^X, then there is a $\hat{\mathfrak{G}} \cong \mathfrak{G}$ such that $D(\hat{\mathfrak{G}}) \leq_T X$.*

The proof of this lemma is the same as Lemma 3.2.7, except we do not copy each component infinitely many times. Now we can state the separation of degree spectra first for repeat daisy graphs and then for 1-1 daisy graphs.

Theorem 3.2.9. *There is a linear order $\mathfrak{L}$ such that* $\mathrm{DgSp}(\mathfrak{L}) \neq \mathrm{DgSp}(\mathfrak{G})$ *for any repeat daisy graph $\mathfrak{G}$.*

Proof. From Lemma 2.2.4, let $\mathfrak{L}$ be a linear ordering such that $\mathfrak{L}$ has no $\mathbf{0}^{(4)}$-copy. From the iterated and relativized version of Theorem 9.10 in Ash and Knight [2] (a result originally due to Watnick [32], as is said in [2]), if $\mathbb{Z}^k \cdot \mathfrak{L}$ has a $\mathbf{0}^{(n)}$-copy for some k and n, then $\mathfrak{L}$ has a $\mathbf{0}^{(n+2k)}$-copy. Let $\mathfrak{A} = \mathbb{Z} \cdot \mathfrak{L}$. Therefore, in our case, $\mathfrak{A}$ has no $\mathbf{0}^{(2)}$ copy. Let $S \subseteq \mathcal{P}(\mathbb{N})$ be a countable family of sets, and let $\mathfrak{G} = \mathfrak{G}_\infty(S)$ be the repeat daisy graph of S.

Assume, for a contradiction, that $\mathrm{DgSp}(\mathfrak{A}) = \mathrm{DgSp}(\mathfrak{G})$. Then every copy of $\mathfrak{A}$ computes a copy of $\mathfrak{G}$. Then Lemma 3.2.6 shows that for every $\tilde{\mathfrak{A}} \cong \mathfrak{A}$, S has an enumeration c.e. in $D(\tilde{\mathfrak{A}})$. By Lemma 3.1.3, $\Diamond_{\mathfrak{A}}^p W_e^D$ enumerates S for some p and e.

Following the proof of Theorem 2.1.14, let $\mathrm{dom}(p)$ consist of $a_0 < \cdots < a_{n-1}$, and for $0 \leq i \leq n$, define the intervals I_i just as we did before. Now associate to each a_i the $l_j \in \mathfrak{L}$ that corresponds to the copy of $\mathbb{Z}$ containing a_i; for each

l_j, give each associated a_i a new name: a_i^j. Say the list of l_j's is $l_1, \ldots, l_m$. For $0 \leq j \leq m$, define intervals L_j in $\mathfrak{L}$ just as we defined I_i in $\mathbb{Z} \cdot \mathfrak{L}$. Notice that, if defined, $I_0 = \mathbb{Z} \cdot L_0 + \mathbb{N}^*$ and $I_n = \mathbb{N} + \mathbb{Z} \cdot L_m$. Notice also that for $0 < i < n$, I_i takes on one of two forms:

$$I_i = k,$$

$$I_i = \mathbb{N} + \mathbb{Z} \cdot L_j + \mathbb{N}^*.$$

By Fact 2.1.12, there are computable orderings K_i such that $K_i \sim_2 L_i$. Replacing L_i by K_i produces a computable ordering $\mathfrak{B}$ with new intervals J_i that replace the I_i. Now by Lemmas 3.2.4 and 3.2.5, $\alpha + \mathbb{Z} \cdot L_j + \beta \sim_3 \alpha + \mathbb{Z} \cdot K_j + \beta$ for any linear orders α and β. Then because the J_i take on one of the forms $J_0 = \mathbb{Z} \cdot K_0 + \mathbb{N}^*$, $J_n = \mathbb{N} + \mathbb{Z} \cdot K_m$, $J_i = k$, and $J_i = \mathbb{N} + \mathbb{Z} \cdot K_j + \mathbb{N}^*$, we have $I_i \sim_3 J_i$. Using Fact 2.1.13, $\mathrm{dom}(p)_{\mathfrak{A}} \sim_3 \mathrm{dom}(p)_{\mathfrak{B}}$. So by Lemma 3.1.5, $\Diamond_{\mathfrak{B}}^q W_e^D$ enumerates S, so S has an enumeration c.e. in $D(\mathfrak{B})'' \equiv_T \emptyset''$ by Lemma 3.1.4. Hence $\mathfrak{G}$ has a $\mathbf{0}^{(2)}$-copy by Lemma 3.2.7. Since $\mathrm{DgSp}(\mathfrak{A}) = \mathrm{DgSp}(\mathfrak{G})$, $\mathfrak{A}$ has a $\mathbf{0}^{(2)}$-copy also, a contradiction. $\square$

Now by applying Lemma 3.2.2, we obtain the following 1-1 enumeration of Theorem 3.2.9.

Theorem 3.2.10. *There is a linear order* $\mathfrak{L}$ *such that* $\mathrm{DgSp}(\mathfrak{L}) \neq \mathrm{DgSp}(\mathfrak{G})$ *for any 1-1 daisy graph* $\mathfrak{G}$.

Proof. Follow the exact same proof as Theorem 3.2.9, only begin with a linear order $\mathfrak{L}$ with no $\mathbf{0}^{(6)}$-copy so that $\mathfrak{A} = \mathbb{Z} \cdot \mathfrak{L}$ has no $\mathbf{0}^{(4)}$-copy. Also, we let $\mathfrak{G} = \mathfrak{G}(S)$, not $\mathfrak{G}_\infty(S)$. Skip then to the end of the proof. Instead of applying Lemma 3.2.7, say that by Lemma 3.2.2, S has a 1-1 enumeration c.e. in $\emptyset^{(4)}$. Hence $\mathfrak{G}$ has a $\mathbf{0}^{(4)}$-copy by Lemma 3.2.8. Again, $\mathrm{DgSp}(\mathfrak{A}) = \mathrm{DgSp}(\mathfrak{G})$, which gives the contradiction. □

Definition 3.2.11. Define the class $\mathcal{E}$ of structures as follows.

- For a countable family $S \subseteq \mathcal{P}(\mathbb{N})$, let $\mathcal{E}_S$ be the class of structures $\mathfrak{A}$ that satisfy the following conditions.
  1. For all $\hat{\mathfrak{A}} \cong \mathfrak{A}$, S has an enumeration in $\Sigma_1^{D(\hat{\mathfrak{A}})}$.
  2. If S has an enumeration in Σ_1^X for some $X \subseteq \mathbb{N}$, then there is an $\hat{\mathfrak{A}} \cong \mathfrak{A}$ such that $D(\hat{\mathfrak{A}}) \leq_T X$.
- Let
$$\mathcal{E} = \bigcup_{S \subseteq \mathcal{P}(\mathbb{N}) \text{ countable}} \mathcal{E}_S.$$

The exact same arguments used in Theorem 3.2.9 yield the following generalized result (where now in place of Lemmas 3.2.6 and 3.2.7, we invoke conditions 1 and 2 in the definition of $\mathcal{E}_S$, respectively).

Theorem 3.2.12. *There is a linear order* $\mathfrak{A}$ *such that* $\mathrm{DgSp}(\mathfrak{A}) \neq \mathrm{DgSp}(\mathfrak{B})$ *for any* $\mathfrak{B} \in \mathcal{E}$.

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