# Research Statement


## Kristen Sellke


An early focus of knot theory was to determine certain characteristics of knots which were invariant under isotopy. To this end, the first polynomial knot invariant, now called the Alexander polynomial, was introduced in the 1920's. About forty years later, Conway developed a method of calculating the Alexander polynomial using skein relations. In 1984 Jones introduced a new polynomial knot invariant based on his work in operator algebras. This discovery led to a renewed interest in polynomial invariants among knot theorists. Today, these polynomials and any associated skein relations are an integral part of knot theory.

The work of Thurston in the 1970's and 80's brought hyperbolic geometry into the study of knots. There is an amazing conjectural equality between a quantity derived from the Jones polynomials of knot cablings and the hyperbolic geometry of the complement of the knot. The $n$th-colored Jones polynomial of a knot $K$, denoted $J_K(n)$, is the Jones polynomial of the knot colored with the $n$-dimensional irreducible representation of $sl(2,\mathbb{C})$. This is equivalent to the Jones polynomial of a linear combination of cablings of the knot. The volume conjecture is as follows: let $K$ be a hyperbolic knot in $S^3$ then

$$\lim_{n\to\infty} \frac{\log\left|J_K(n)(e^{\frac{2\pi i}{n}})\right|}{n} = \frac{1}{2\pi}\mathrm{vol}(S^3-K),$$

where $\mathrm{vol}(S^3-K)$ is the volume in the knot complement of a complete hyperbolic metric. Using an approach from complex analysis involving holonomic functions, Garoufalidis and Le showed that the colored Jones polynomials satisfy a recurrence relations in terms of the variable $n$ and hence the limit converges.

One aspect of my research is involved with finding this recurrence relation through ideas from quantum topology. The basic building block of this approach is an invariant of 3-manifolds called the Kauffman bracket skein module (KBSM). There is a skein relation, due to Kauffman, which can be used to calculate the Jones polynomial of a knot. Przytycki applied this skein relation to isotopy classes of framed links in 3-manifolds. By doing this he introduced an invariant of 3-manifolds, now called the Kauffman bracket skein module. This invariant is a quotient of a free module over a submodule given by the Kauffman skein relations. If we let $\mathcal{R}$ be the ring of Laurent polynomials, $\mathbb{C}[t,t^{-1}]$, then our free module is the free $\mathcal{R}$-module with basis all framed links in the 3-manifold considered up to isotopy.

The next building block is an extension of yet another polynomial knot invariant from the early 1990's, called the A-polynomial. This two-variable polynomial is determined by looking at the representation space of the fundamental group of a knot complement in $SL(2, \mathbb{C})$. Frohman, Gelca and LoFaro used a pairing of the KBSM of a knot complement and the KBSM of a cylinder over a torus, given by gluing the cylinder over a torus to the boundary of the knot complement, to construct a knot invariant they called the non-commutative A-ideal. By specializing the non-commutative A-ideal to $t = -1$, we are able to recover the classic A-polynomial. In addition, if we can show that this ideal is non-trivial, then the colored Jones polynomials $J_K(n)$ satisfy a recurrence relation in terms of $n$.

At present, I am working on applying the above framework to $(p, q)$ torus knots. The first step has been to find a presentation for the Kauffman bracket skein module of the complement of a torus knot. The complement of a torus knot is a handlebody of genus 2 with a two-handle attached. Hoste and Przytycki have shown the KBSM of such a 3-manifold has a presentation with generators given by generators of the KBSM of the handlebody and with relations given by all possible handleslides of the generators across the attaching curve. The presentation of the KBSM of the knot complement is then simplified by using the handleslides to kill off as many generators as possible.

This method has been successfully used to calculate the KBSM of a few families of knot complements. Bullock proved that the KBSM of the complement of a $(2, 2p + 1)$ torus knot is a free $\mathbb{C}[t, t^{-1}]$module with basis $\{x^n y^m, \mathcal{U}\}$ where $0 \leq m \leq p$, $x$ is a meridian curve, $y$ is a representative of a certain framed link isotopy class and $\mathcal{U}$ is the unknot. In addition, it has recently been shown that the KBSM of two other knot families is a free $\mathbb{C}[t, t^{-1}]$ module. Bullock and LoFaro proved this for twist knots and presented a basis and Le did the same for 2-bridge knots.

After finding the KBSM of a torus knot, my present research goal is to show that the non-commutative A-ideal is non-trivial and hence give an alternate method for finding the recurrence relations for the colored Jones polynomials of a torus knot. Next, I would like to use the computer programming and modeling skills I have learned in this process to calculate the Kauffman bracket skein module for a knot complement where the handlebody involved is of genus 3.

The last decade has seen great growth in the discovery and application of these computational algorithms in 3-dimensions. In the future, I am interested in investigating the implications that these algorithms may have on quantum invariants of 3-manifolds.