MATH 5210Q   Abstract Algebra I   Fall 2009

Assignment 1: Group Theory

You can work on this assignment in groups. If you do so, please indicate on this page the names of people you have worked with. Everybody needs to hand in his/her own copy in class on October 1.

Name : ________________________________

Other group members : ____

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Problem	*Points*	*Score*
1	2	
2	10	
3	10	
4	10	
5	8	
Total	40	

1. Let G be an abelian group, and define $\text{Tor}(G) = \{g \in G \mid |g| < \infty\}$. Prove that $\text{Tor}(G)$ is a subgroup of G. ($\text{Tor}(G)$ is called the torsion subgroup).

2. Let F be a field and define

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in F \right\}.$$

   (a) Prove that G is a subgroup of $Sl_n(F)$.

   (b) Prove that the center $Z(G)$ of G is isomorphic to $(F, +)$.

   (c) Prove that the quotient $G/Z(G)$ is isomorphic to $(F^2, +)$.

   (d) Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. Prove that for all $k \geq 1$,

$$A^k = \begin{bmatrix} 1 & ak & bk + ac\frac{k(k-1)}{2} \\ 0 & 1 & ck \\ 0 & 0 & 1 \end{bmatrix}$$

   (e) Let $F = \mathbb{Z}/p\mathbb{Z}$ with p a prime number. Compute the order of the element

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Let G be a group and let $\text{Aut}(G)$ be the group of automorphisms of G. Thus the elements of $\text{Aut}(G)$ are isomorphisms $G \to G$, and the group operation is the composition of maps. For $g \in G$, let $\phi_g : G \to G$ be the map defined by $\phi_g(x) = gxg^{-1}$ for all $x \in G$ and let $\text{Inn}(G) = \{\phi_g \mid g \in G\}$.

   (a) Prove that $\phi_g \in \text{Aut}(G)$ for all $g \in G$.

   (b) Prove that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$. ($\text{Inn}(G)$ is called the group of inner automorphisms of G.)

   (c) Prove that $\text{Inn}(G)$ is isomorphic to $G/Z(G)$, where $Z(G)$ denotes the center of G.

   (d) Now let G be the dihedral group D_8. Prove that $\text{Inn}(G)$ is isomorphic to the Klein four group V_4.

4. Let Z_n be a cyclic group of order $n < \infty$. For any $a \in \mathbb{Z}$ define $\sigma_a : Z_n \to Z_n$ by $\sigma_a(x) = x^a$ for all $x \in Z_n$. Prove that

   (a) σ_a is an automorphism of Z_n if and only if $(a, n) = 1$,

   (b) $\sigma_a = \sigma_b$ if and only if $a \equiv b \pmod{n}$,

   (c) any automorphism of Z_n is of the form σ_a for some $a \in \mathbb{Z}$,

   (d) the group of automorphisms $\text{Aut}(Z_n)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$

5. Find all normal subgroups N of D_8 and for each of them, find the isomorphism type of the quotient D_8/N.