MATH 5210Q Abstract Algebra I Fall 2009

Assignment 2: Group Theory

You can work on this assignment in groups. If you do so, please indicate on this page the names of people you have worked with. Everybody needs to hand in his/her own copy in class on November 5.

Name : ________________________________

Other group members : ________________________________

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Problem	*Points*	*Score*
1	6	
2	8	
3	4	
4	8	
5	4	
6	2	
7	8	
Total	40	

1. Let V_4 denote the Klein 4-group.

   (a) Show that there exists an isomorphism $f : \mathrm{Aut}(V_4) \to S_3$.

   (b) Show that the symmetric group S_4 is isomorphic to a semi-direct product $V_4 \ltimes_\varphi S_3$.

2. (a) Show that $\mathrm{Aut}(Z_8)$ is isomorphic to V_4.

   (b) Show that there are exactly 4 distinct homomorphisms from Z_2 to $\mathrm{Aut}(Z_8)$.

   (c) Show that the four homomorphisms in part (b) give rise to four non-isomorphic semi-direct products $Z_8 \ltimes_\varphi Z_2$.

   (d) Prove that $Z_8 \times Z_2$ and D_{16} are two of the groups in (c).

3. Show that the alternating group A_4 is a semi-direct product of V_4 and Z_3.

4. Let G be a group and let G' be the commutator subgroup of G. Recall that G' is generated by all commutators $x^{-1}y^{-1}xy$, with $x, y \in G$. Prove that

   (a) $G' \lhd G$.

   (b) G/G' is abelian.

   (c) If $H \lhd G$ such that G/H is abelian, then $G' \leq H$.

   (d) If $H \leq G$ such that $G' \leq H$, then $H \lhd G$.

5. Prove that subgroups of solvable groups are solvable.

6. Let F be a field and define

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in F \right\}.$$

   Show that G is solvable.

7. Burnside's formula. Let G be a finite group acting on a finite set X. For any $a \in X$, denote by $\mathcal{O}_a$ the orbit of a, and let N be the number of orbits of this action. For $g \in G$, let $X_g = \{x \in X \mid g \cdot x = x\}$ be the set of points in X that are fixed by the action of g.

   (a) Prove that

$$\sum_{g \in G} |X_g| = |G| \sum_{a \in X} \frac{1}{|\mathcal{O}_a|}.$$

   (b) Prove that

$$\sum_{a \in X} \frac{1}{|\mathcal{O}_a|} = N.$$

   (c) Deduce Burnside's formula for the number of orbits:

$$N = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$