Final Exam Preparation

1. Let $V = \mathbb{F}^n$ and let $B = (v_1, v_2, \ldots, v_n)$ be an orthonormal basis of $V$. Let $S_i : V \to V$ be the map that sends a vector $v = a_1 v_1 + \cdots + a_n v_n$ to $S_i(v) = v - 2a_i v_i$.
   
   (a) Show that $S_i$ is linear.
   (b) Show that $S_i$ is an isometry.

2. Let $T_C : \mathbb{C} \to \mathbb{C}$ be the map given by complex conjugation, that is, $T_C(a + ib) = T_C(a - ib)$, and let $T_R : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $T_R(a, b) = (a, -b)$.
   
   (a) Show that $T_C$ is not a linear map.
   (b) Show that $T_R$ is a linear map.
   (c) Show that $T_R$ is an isometry but not positive.

3. Let $\mathcal{S}$ be the set of all self-adjoint operators on a vector space $V$.
   
   (a) Prove that $\mathcal{S}$ is a subspace of $\mathcal{L}(V)$ if $\mathbb{F} = \mathbb{R}$.
   (b) Prove that $\mathcal{S}$ is not a subspace of $\mathcal{L}(V)$ if $\mathbb{F} = \mathbb{C}$.

4. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be the reflection at the line $\{(x, x) \mid x \in \mathbb{R}\}$.
   
   (a) Find the matrix of $T$ with respect to the standard basis.
   (b) Find all eigenvalues of $T$.

5. Let $V$ be a complex vector space and let $T \in \mathcal{L}(V)$ be normal and such that $T$ has only one distinct eigenvalue $\lambda$. Show that $Tv = \lambda v$ for all $v \in V$.

6. Let $V$ be real vector space and let $T \in \mathcal{L}(V)$ be an isometry. Prove that $\lambda = 0$ can not be an eigenvalue of $T$.

7. (a) State the spectral theorem.
   (b) When is an operator said to be positive. Give a precise definition.
   (c) Give an example of a positive operator on $\mathbb{R}^2$ that is not the identity.
   (d) Does every complex positive operator have a diagonal matrix with respect to some orthonormal basis? Explain why or why not.

8. Let $T$ be a self-adjoint operator. Show that $\text{Null} T = (\text{Range} T)^\perp$. 

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