Course Announcement for Spring 2003
Topics in Geometry: The Yamabe Problem

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The Yamabe problem is the following: if someone hands you a compact Riemannian manifold (M, g_0), can you find a metric g which is conformal to g_0 and has constant scalar curvature? The answer turns out to be "yes", but the proof took about 25 years to develop.

Surprisingly, the hard case is that of the sphere. One can see this difficulty when examining how scalar curvature transforms under a conformal change of the metric. If $g = u^{\frac{4}{n-2}} g_0$, $u > 0$ then

$$\Delta_{g_0} u - \left(\frac{n-2}{4(n-1)} \right) R_{g_0} u + \left(\frac{n-2}{4(n-1)} \right) R_g u^{\frac{n+2}{n-2}} = 0, \tag{1}$$

where R_{g_0} and R_g are the scalar curvatures of g_0 and g respectively. If one takes g_0 to be the flat metric on $\mathbb{R}^n$, identified with $S^n \backslash \{p\}$, and sets $R_g = n(n-1)$, then equation (1) becomes

$$\Delta u + \left(\frac{n(n-2)}{4} \right) u^{\frac{n+2}{n-2}} = 0 \qquad u : \mathbb{R}^n \to (0, \infty),$$

which is a semilinear equation with a critical nonlinearity. This problem has many interesting connections with analysis and topology, including finding a sharp Sobolev constant and restrictions on the topology of manifolds with positive scalar curvature.

I plan to start by following the survey article of Lee and Parker (Bull. Amer. Math. Soc. 17: 37–91, 1987), which I will distribute, and give a (more or less) complete proof that the Yamabe problem always has a solution. If time permits, I might discuss the singular Yamabe problem, where one allows singular solutions to equation (1).

Strictly speaking, the prerequisite for this class should be an understanding of basic Riemannian geometry, like what is covered in Math 6170. However, talk to me if you don't have this background and wish to take this course; I would gladly point you in the right reference direction. Also, Prof. Hartenstine's course in the fall will help you appreciate some of the analytical subtleties of this problem (but it is not a prerequisite).