An Example

Many people seemed confused by the last example we did in class on Nov. 20. So here are some more details. Recall that we were solving the differential equation

$$\partial_t^2 u = \partial_x^2 u$$

with the initial conditions

$$u(x,0) = f(x) = 0, \qquad \partial_t u(x,0) = g(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

The If we take the Fourier transform of this equation in the x variable, we obtain the differential equation

$$\hat{u}''(\omega,t) = -\omega^2 \hat{u}(\omega,t),$$

which is a one-parameter family of ODEs. Here the $'$ denotes differentiation with respect to t. The general solution of the equation for $\hat{u}$ is then

$$\hat{u}(\omega,t) = \hat{f}(\omega)\cos(\omega t) + \frac{\hat{g}(\omega)}{\omega}\sin(\omega t).$$

Taking the inverse Fourier transform, we have

$$u(x,t) = \mathcal{F}^{-1}(\hat{f}(\omega)\cos(\omega t) + \frac{\hat{g}(\omega)}{\omega}\sin(\omega t)).$$

So far, this has all been standard, but now we have to plug in our initial conditions: $f(x) = 0$, so $\hat{f}(\omega) = 0$. The initial velocity g is a square pulse, so we have

$$\hat{g}(\omega) = \frac{2\sin(\omega)}{\sqrt{2\pi}\omega} = \sqrt{2/\pi}\frac{\sin(\omega)}{\omega}.$$

So

$$\hat{u}(\omega,t) = \sqrt{2/pi}\frac{\sin(\omega)\sin(\omega t)}{\omega^2} = \sqrt{2/\pi}\left(\frac{\sin(\omega)}{\omega}\right)\left(\frac{\sin(\omega t)}{\omega}\right).$$

We want to recognize $\hat{u}$ as the Fourier transform of something. The key observation is that it's pretty close; indeed, the first factor in the product is the Fourier transform of the initial square pulse. The second factor in the product looks very similar; if we can manipulate it a little bit then we're basically done, becasue the product of two Fourier transforms in the Fourier transform of a convolution. So let's look at the second term:

$$g^* := \frac{\sin(\omega t)}{\omega}.$$

The factor of t in the sin part of g^* fouls everything up, so let's see if we can get rid of it by looking at the inverse Fourier transform. We will use the change of variables $\eta = \omega t$ below. Then $\omega = \eta/t$ and $d\omega = d\eta/t$. If we compute the inverse Fourier transform of g^* we find

$$\begin{aligned}
\check{g}^*(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\omega} g^*(\omega,t)d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\omega}\frac{\sin(\omega t)}{\omega}d\omega \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\eta/t}\frac{\sin(\eta)}{\eta/t}d\eta/t = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(x/t)\eta}\frac{\sin(\eta)}{\eta}d\eta \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(x/t)\eta}\sqrt{\pi/2}\hat{g}(\eta)d\eta = \sqrt{\pi/2}g(x/t)
\end{aligned}$$

Thus if we let $g_t(x) := g(x/t)$, we have

$$g^* = \frac{\sin(\omega t)}{\omega} = \sqrt{\pi/2}g_t(x) = \sqrt{\pi/2}g(x/t).$$

Now we plug this into our solution. We have

$$\hat{u}(\omega, t) = \sqrt{2/\pi}\left(\frac{\sin(\omega)}{\omega}\right)\left(\frac{\sin(\omega t)}{\omega}\right) = \frac{1}{2}\hat{g}(\omega)\hat{g}_t(\omega) = \frac{1}{2}\mathcal{F}(g * g_t)(\omega).$$

Taking the inverse Fourier transform we have

$$u(x,t) = \frac{1}{2}g * g_t(x).$$

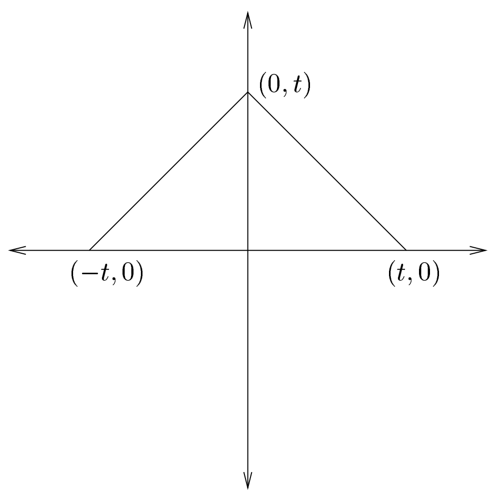
We can actually compute this convolution:

$$g * g_t * x) = \int_{-\infty}^{\infty} g_t(x-y)g(y)dy = \int_{-1}^{1} g_t(x-y)dy = \int_{-1}^{1} g((x-y)/t)dy.$$

This last integral just measures t times the length of the overlap of the intervals $[-1,1]$ and $[(y-1)/t, (y+1)/t]$. Measuring this overlap, we see

$$g * g_t(x) = \begin{cases} t + x & -t \leq x \leq 0 \\ t - x & 0 \leq x \leq t \\ 0 & \text{otherwise.} \end{cases}$$

Thus the waveform at time t looks like



There are two final comments. First: the wave form should look sort of like that. The initial position was zero, and the initial velocity was a non-negative, even function. So the wavefronts should push out symmetrically from the origin. Think of what happens when you throw a pebble into a still pond. Second: how did I come up with this crazy formula? Well, when I was trying to come up with example of for class I first wrote down the case where the initial velocity is zero and the initial position is the square pulse. When I did that, I noticed that the Fourier transform of the square pulse looked a lot like the $(1/\omega)\sin(\omega t)$ term, which every solution to the wave equation has. That hinted at the change of variables above.