Practice Problems

Math 3150

November 21, 2003

These are only practice problems for the material in chapter 7. You might also want to make sure you know how to do the problem on the earlier practice sheets.

1. Compute the Fourier transform of the following functions.

   (a) $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

   (b) $f(x) = \begin{cases} -x & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$

   (c) $f(x) = \begin{cases} 1 + x & -1 \leq x \leq 1 \\ 1 - x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

   (d) $f(x) = e^{-|x|}$

2. Let $f = f(x)$ and $g = g(x)$ be piece-wise continuous, real-valued functions such that $f(x) = 0$ for $|x| > 1$ and $g(x) = 0$ for $|x| > 1$. In particular, this implies both f and g are bounded and achieves their maxima.

   (a) Show that the Fourier transform $\hat{f}$ makes sense (i.e. the improper integral converges) and $|\hat{f}(\omega)| \leq \max_x |f(x)|$ for all ω.

   (b) Is it true that $\hat{f}(\omega) = 0$ for $|\omega|$ sufficiently large? Explain your answer.

   (c) Is it true that $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ is zero for $|x|$ sufficiently large? Explain your answer.

   (d) If the Fourier transform $\hat{f}$ is also real-valued, what can you say about f?

   (e) Show that $\mathcal{F}(f * g)(\omega) = \sqrt{2\pi}\hat{f}(\omega)\hat{g}(\omega)$.

3. Solve the wave equation

   $$\partial_t^2 u = \partial_x^2 u$$

   with the initial conditions

   $$u(x,t) = f(x) = 0, \qquad \partial_t u(x,0) = \frac{1}{1+x^2}.$$

   You may leave your answer in terms of the inverse Fourier transform (even though it is possible to simplify it more).

4. Solve the differential equation

   $$\partial_t u = \frac{1}{1+t}\partial_x^2 u$$

   for $t \geq 0$, with the initial condition

   $$u(x,0) = f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

   You may leave your answer in terms of the inverse Fourier transform.

5. Solve the differential equation

   $$\partial_t u = \partial_x^2 u + u$$

   for $t \geq 0$, with the initial conditions

   $$u(x,0) = f(x) = \begin{cases} x + 1 & -1 \leq x \leq 0 \\ 1 - x & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   You may leave your answer in terms of the inverse Fourier transform.

6. Solve the heat equation

$$\partial_t u = \partial_x^2 u$$

for $t \geq 0$, with the initial condition

$$u(x,0) = f(x) = e^{-x^2/2}.$$

You may leave your answer in terms of a convolution with the heat kernel (even though it is possible to simplify it more).

7. Solve the heat equation

$$\partial_t u = \partial_x^2 u$$

for $t \geq 0$, with the initial condition

$$u(x,0) = f(x) = \begin{matrix} 1 & |x| < 1 \\ 0 & |x| \geq 1. \end{matrix}$$

You may leave your answer in terms of a convolution with the heat kernel (even though it is possible to simplify it more).

8. Let u be a solution to the heat equation

$$\partial_t = \partial_x^2 u$$

on a line, with $u(x,t) \to 0$ very rapidly as $|x| \to \infty$. For instance, a sufficient decay condition is

$$|u(x,t)| \leq \frac{1}{1+x^2}$$

for all x and $t \geq 0$.

(a) Define the energy $E(t)$ by

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u(x,t))^2 dx$$

and show that E is a decreasing function of t. (Hint: use the equation and integrate by parts!)

(b) Use the condition on E to say what happens to $u(x,t)$ as $t \to \infty$. You might want to also think about this physically.