

Solutions to the Practice Problems
Math 3150
November 25, 2003

These are only practice problems for the material in chapter 7. You might also want to make sure you know how to do the problem on the earlier practice sheets.

1. Compute the Fourier transform of the following functions.

$$(a) f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\omega} dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ix\omega} dx - \frac{1}{\sqrt{2\pi}} \int_{-1}^0 e^{-ix\omega} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{i\omega} e^{-ix\omega} \Big|_{x=0}^{x=1} + \frac{1}{i\omega} e^{-ix\omega} \Big|_{x=-1}^{x=0} \right] = \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-i\omega}}{i\omega} + \frac{1}{i\omega} + \frac{1}{i\omega} - \frac{e^{i\omega}}{i\omega} \right] \\ &= \frac{1}{i\omega\sqrt{2\pi}} [2 - e^{i\omega} - e^{-i\omega}] = \frac{2 - 2\cos\omega}{i\omega\sqrt{2\pi}}. \end{aligned}$$

$$(b) f(x) = \begin{cases} -x & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 x e^{-ix\omega} dx = \frac{1}{\sqrt{2\pi}} \left[-\frac{x}{i\omega} e^{-ix\omega} \Big|_{x=-1}^{x=1} + \frac{1}{i\omega} \int_{-1}^1 e^{-ix\omega} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-i\omega}}{i\omega} - \frac{e^{i\omega}}{i\omega} + \frac{1}{\omega^2} e^{-ix\omega} \Big|_{x=-1}^{x=1} \right] = \frac{1}{\sqrt{2\pi}} \left[-\frac{2\cos\omega}{i\omega} + \frac{e^{-i\omega} - e^{i\omega}}{\omega^2} \right] = \frac{1}{\sqrt{2\pi}} \left[\frac{2i\cos(\omega)}{\omega} + \frac{2\sin(\omega)}{\omega^2} \right] \\ &= \frac{i\omega^2\sqrt{2}}{\sqrt{\pi}} [\omega\cos(\omega) + \sin(\omega)] \end{aligned}$$

$$(c) f(x) = \begin{cases} 1+x & -1 \leq x \leq 1 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

First let's do some preliminary computations:

$$\begin{aligned} \int_0^1 e^{-ix\omega} dx &= -\frac{1}{i\omega} e^{-ix\omega} \Big|_0^1 = \frac{1}{i\omega} [1 - e^{-i\omega}] \\ \int_{-1}^0 e^{-ix\omega} dx &= -\frac{1}{i\omega} e^{-ix\omega} \Big|_{-1}^0 = \frac{1}{i\omega} [e^{i\omega} - 1] \\ \int_0^1 -x e^{-ix\omega} dx &= \frac{x}{i\omega} e^{-ix\omega} \Big|_0^1 - \frac{1}{i\omega} \int_0^1 e^{-ix\omega} dx = \frac{e^{-i\omega}}{i\omega} - \frac{1}{\omega^2} e^{-ix\omega} \Big|_0^1 = \frac{e^{-i\omega}}{i\omega} + \frac{1 - e^{-i\omega}}{\omega^2} \\ \int_{-1}^0 x e^{-ix\omega} dx &= -\frac{x}{i\omega} e^{-ix\omega} \Big|_{-1}^0 + \frac{1}{\omega^2} \int_{-1}^0 e^{-ix\omega} dx = -\frac{e^{i\omega}}{i\omega} + \frac{1}{\omega^2} e^{-ix\omega} \Big|_{-1}^0 = -\frac{e^{i\omega}}{i\omega} + \frac{1}{\omega^2} [1 - e^{i\omega}]. \end{aligned}$$

Now put everything together:

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\omega} dx = \frac{1}{\sqrt{2\pi}} \int_0^1 (1-x)e^{-ix\omega} dx + \frac{1}{\sqrt{2\pi}} \int_{-1}^0 (1+x)e^{-ix\omega} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ix\omega} dx + \frac{1}{\sqrt{2\pi}} \int_{-1}^0 e^{-ix\omega} dx + \frac{1}{\sqrt{2\pi}} \int_0^1 -x e^{-ix\omega} dx + \frac{1}{\sqrt{2\pi}} \int_{-1}^0 x e^{-ix\omega} dx \\ &= \frac{1}{i\omega\sqrt{2\pi}} [1 - e^{-i\omega}] + \frac{1}{i\omega\sqrt{2\pi}} [e^{i\omega} - 1] + \frac{e^{-i\omega}}{i\omega\sqrt{2\pi}} + \frac{1}{\omega^2\sqrt{2\pi}} (1 - e^{-i\omega}) + \frac{-e^{i\omega}}{i\omega\sqrt{2\pi}} + \frac{1}{\omega^2\sqrt{2\pi}} (1 - e^{i\omega}) \\ &= \frac{1}{\omega^2\sqrt{2\pi}} (2 - e^{i\omega} - e^{-i\omega}) = \frac{2 - 2\cos\omega}{\omega^2\sqrt{2\pi}} = \sqrt{2/\pi} \frac{1 - \cos\omega}{\omega^2}. \end{aligned}$$

(d) $f(x) = e^{-|x|}$

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\omega} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-ix\omega} e^x dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ix\omega} e^{-x} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{x(1-i\omega)} dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{x(-1-i\omega)} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\omega} e^{x(1-i\omega)} \Big|_{-\infty}^0 - \frac{1}{1+i\omega} e^{x(-1-i\omega)} \Big|_0^{\infty} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-i\omega} + \frac{1}{1+i\omega} \right] = \frac{1}{\sqrt{2\pi}} \cdot \frac{1+i\omega+1-i\omega}{(1-i\omega)(1+i\omega)} = \sqrt{2/\pi} \frac{1}{1+\omega^2}\end{aligned}$$

2. Let $f = f(x)$ and $g = g(x)$ be piece-wise continuous, real-valued functions such that $f(x) = 0$ for $|x| > 1$ and $g(x) = 0$ for $|x| > 1$. In particular, this implies both f and g are bounded and achieves their maxima.

(a) Show that the Fourier transform \hat{f} makes sense (i.e. the improper integral converges) and $|\hat{f}(\omega)| \leq \max_x |f(x)|$ for all ω .

The improper integral is in fact a proper integral, from $x = -1$ to $x = 1$. This is because $f(x) = 0$ for $|x| > 1$. We also have the bound

$$\begin{aligned}|\hat{f}(\omega)| &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |e^{-ix\omega}| |f(x)| dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 |f(x)| dx \leq \frac{\max |f|}{\sqrt{2\pi}} \int_{-1}^1 dx \\ &= \frac{2}{\sqrt{2\pi}} \max |f| \leq \max |f|.\end{aligned}$$

The last inequality comes from $\sqrt{2\pi} > 2$.

(b) Is it true that $\hat{f}(\omega) = 0$ for $|\omega|$ sufficiently large? Explain your answer.

No. For instance, if f is a square pulse (i.e. $f(x) = 1$ for $|x| \leq 1$ and $f(x) = 0$ for $|x| > 1$), then the Fourier transform is

$$\hat{f}(\omega) = \sqrt{2/\pi} \frac{\sin(\omega)}{\omega},$$

which is only zero at $\omega = \pm\pi, \pm2\pi, \pm3\pi, \dots$

(c) Is it true that $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ is zero for $|x|$ sufficiently large? Explain your answer.

Yes. Indeed, if $|y| > 1$ and $|x| > 2$ then both $f(x-y) = 0$ and $g(y) = 0$, and so the integrand in the convolution is zero.

(d) If the Fourier transform \hat{f} is also real-valued, what can you say about f ?

Recall that

$$\begin{aligned}\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\cos(x\omega) - i \sin(x\omega)) f(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(x\omega) f(x) dx - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(x\omega) f(x) dx.\end{aligned}$$

The first integral picks out the even part of f and the second integral picks out the odd part. Therefore, if $\hat{f}(\omega)$ is real valued, the second integral must be zero for all ω , and so f must be an even function.

(e) Show that $\mathcal{F}(f * g)(\omega) = \sqrt{2\pi} \hat{f}(\omega) \hat{g}(\omega)$.

$$\begin{aligned}\mathcal{F}(f * g)(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f * g(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} \int_{-\infty}^{\infty} f(x-y)g(y) dy dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\omega} f(x-y)g(y) dx dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ix\omega} e^{-iy\omega} e^{iy\omega} f(x-y)g(y) dx dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(x-y)} f(x-y) d(x-y) \cdot \int_{-\infty}^{\infty} e^{-iy\omega} g(y) dy \\ &= \sqrt{2\pi} \hat{f}(\omega) \cdot \hat{g}(\omega).\end{aligned}$$

3. Solve the wave equation

$$\partial_t^2 u = \partial_x^2 u$$

with the initial conditions

$$u(x, 0) = f(x) = 0, \quad \partial_t u(x, 0) = \frac{1}{1+x^2}.$$

You may leave your answer in terms of the inverse Fourier transform (even though it is possible to simplify it more).

Let

$$g(x) = \frac{1}{1+x^2} = \partial_t u(x, 0).$$

Take the Fourier transform of each side of the equation in the x variable. Then we get

$$\hat{u}''(\omega, t) = -\omega \hat{u}(\omega, t), \quad \hat{u}(\omega, 0) = 0, \quad \hat{u}'(\omega, 0) = \hat{g}(\omega).$$

This has the solution

$$\hat{u}(\omega, t) = \frac{\hat{g}(\omega)}{\omega} \sin(\omega t),$$

and so

$$u(x, t) = \mathcal{F}^{-1} \left[\frac{\hat{g}(\omega) \sin(\omega t)}{\omega} \right].$$

4. Solve the differential equation

$$\partial_t u = \frac{1}{1+t} \partial_x^2 u$$

for $t \geq 0$, with the initial condition

$$u(x, 0) = f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

You may leave your answer in terms of the inverse Fourier transform.

Again, take the Fourier transform in x . We get

$$\hat{u}' = -\frac{\omega^2}{1+t} \hat{u}, \quad \hat{u}(\omega, 0) = \hat{f}(\omega) = \sqrt{2/\pi} \frac{\sin \omega}{\omega}.$$

This is a separable, first order ODE, so

$$\frac{d\hat{u}}{\hat{u}} = -\omega^2 \frac{dt}{1+t},$$

which we can integrate to get

$$\ln \hat{u} = -\omega^2 \ln(1+t) + c = \ln[c(1+t)^{-\omega^2}].$$

Thus

$$\hat{u}(\omega, t) = c(1+t)^{-\omega^2} = \hat{u}(\omega, 0)(1+t)^{-\omega^2} = \sqrt{2/\pi} \frac{\sin \omega}{\omega} (1+t)^{-\omega^2}.$$

Taking the inverse Fourier transform, we have

$$u(x, t) = \sqrt{2/\pi} \mathcal{F}^{-1} \left[\frac{\sin \omega}{\omega} (1+t)^{-\omega^2} \right].$$

5. Solve the differential equation

$$\partial_t u = \partial_x^2 u + u$$

for $t \geq 0$, with the initial conditions

$$u(x, 0) = f(x) = \begin{cases} x+1 & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

You may leave your answer in terms of the inverse Fourier transform.

Taking the Fourier transform in x , we get

$$\hat{u}' = -\omega^2 \hat{u} + \hat{u} = (1-\omega^2) \hat{u}, \quad \hat{u}(\omega, 0) = \hat{f}(\omega) = \sqrt{2/\pi} \frac{1-\cos \omega}{\omega^2}.$$

This ODE has the solution

$$\hat{u}(\omega, t) = \hat{u}(\omega, 0)e^{(1-\omega^2)t} = \hat{f}(\omega)e^{(1-\omega^2)t} = \sqrt{2/\pi} \frac{e^{(1-\omega^2)t}(1 - \cos \omega)}{\omega^2}.$$

Taking the inverse Fourier transform, we have

$$u(x, t) = \sqrt{2/\pi} \mathcal{F}^{-1} \left[\frac{e^{(1-\omega^2)t}(1 - \cos \omega)}{\omega^2} \right].$$

6. Solve the heat equation

$$\partial_t u = \partial_x^2 u$$

for $t \geq 0$, with the initial condition

$$u(x, 0) = f(x) = e^{-x^2/2}.$$

You may leave your answer in terms of a convolution with the heat kernel (even though it is possible to simplify it more).

The solution to the heat equation is given by

$$\begin{aligned} u(x, t) &= ? * f(x, y) \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} ?(x-y, t) f(y) dy = \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} f(y) dy \\ &= \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)} e^{-y^2/2} dy \end{aligned}$$

There is another way to look at the solution. Recall that the solution to the transformed heat equation is a Gaussian. Indeed, the transformed equation (after the Fourier transform) is

$$\hat{u}'' = -\omega^2 \hat{u}, \quad \hat{u}(\omega, 0) = \hat{f}(\omega),$$

which has the solution

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-\omega^2 t} = \hat{f}(\omega) e^{-(\sqrt{2t}\omega)^2/2} = e^{-\omega^2/2} \cdot e^{-(\sqrt{2t}\omega)^2/2}.$$

We have used the fact that in this case f is a Gaussian, so $\hat{f}(\omega) = f(\omega)$, and \hat{u} is a product of Gaussians.

There is a general rule hiding here. Indeed, if $g(x) = e^{-ax^2/2}$ and $h(x) = e^{-bx^2/2}$ for two positive numbers a and b , then

$$\hat{g}(\omega) = \frac{1}{\sqrt{a}} e^{-\omega^2/(2a)}, \quad \hat{h}(\omega) = \frac{1}{\sqrt{b}} e^{-\omega^2/(2b)},$$

which implies that

$$\mathcal{F}(g * h)(\omega) = \sqrt{2\pi} \hat{g}(\omega) \hat{h}(\omega) = \sqrt{\frac{2\pi}{ab}} e^{-\omega^2/2a} e^{-\omega^2/2b} = \sqrt{\frac{2\pi}{ab}} e^{-\omega^2(2ab)/(a+b)}.$$

However, the right hand side of the equation above is the Fourier transform of

$$\sqrt{\frac{2\pi}{a+b}} e^{-x^2 ab/(2(a+b))},$$

which means (after taking the inverse Fourier transform) that

$$g * h(x) = \sqrt{\frac{2\pi}{a+b}} e^{-x^2 ab/(2(a+b))}.$$

Now we can use this to find an explicit formula for the $u(x, t) = \mathcal{F}^{-1}(\hat{u})$. Indeed, if $f_t(x) = \sqrt{2t} f(x/\sqrt{2t})$, then $\hat{f}_t(\omega) = e^{-\omega^2 t}$ and

$$\hat{u}(\omega, t) = \hat{f}(\omega) \cdot \hat{f}_t(\omega) = \mathcal{F}(f * f_t)(\omega).$$

Taking the inverse Fourier transform, we have

$$u(x, t) = f * f_t(x) = \sqrt{\frac{2\pi}{1+2t}} e^{-x^2/(2t+1)}.$$

Here we have used the formula we derived for the convolution of two Gaussians.

7. Solve the heat equation

$$\partial_t u = \partial_x^2 u$$

for $t \geq 0$, with the initial condition

$$u(x, 0) = f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

You may leave your answer in terms of a convolution with the heat kernel (even though it is possible to simplify it more).

$$\begin{aligned} u(x, t) &= ? * f(x, t) = \frac{1}{\sqrt{2t}} \int_{-\infty}^{\infty} ?(x-y, t) f(y) dy = \frac{1}{\sqrt{2t}} \int_{-1}^1 e^{-(x-y)^2/(4t)} dy \\ &= \frac{1}{\sqrt{2t}} \int_{x-1}^{x+1} e^{-z^2/(4t)} dz = \sqrt{2} \int_{(x-1)/(2\sqrt{t})}^{(x+1)/(2\sqrt{t})} e^{-w^2} dw \\ &= \sqrt{2} [\operatorname{erf}((x+1)/(2\sqrt{t})) - \operatorname{erf}((x-1)/(2\sqrt{t}))] \end{aligned}$$

8. Let u be a solution to the heat equation

$$\partial_t u = \partial_x^2 u$$

on a line, with $u(x, t) \rightarrow 0$ very rapidly as $|x| \rightarrow \infty$. For instance, a sufficient decay condition is

$$|u(x, t)| \leq \frac{1}{1+x^2}$$

for all x and $t \geq 0$.

(a) Define the energy $E(t)$ by

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} (u(x, t))^2 dx$$

and show that E is a decreasing function of t . (Hint: use the equation and integrate by parts!)

To show E is decreasing, we will show that $E' < 0$. First observe that u is not identically zero, and it decays for large x . This means $\partial_x u$ cannot be identically zero. We have

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \int_{-\infty}^{\infty} u^2(x, t) dx = \int_{-\infty}^{\infty} \partial_t u^2(x, t) dx = \int_{-\infty}^{\infty} 2u(x, t) \partial_t u(x, t) dx \\ &= 2 \int_{-\infty}^{\infty} u(x, t) \partial_x^2 u(x, t) dx = 2 u \partial_x u \Big|_{x=-\infty}^{x=\infty} - 2 \int_{-\infty}^{\infty} (\partial_x u(x, t))^2 dx = -2 \int_{-\infty}^{\infty} (\partial_x u(x, t))^2 dx < 0. \end{aligned}$$

In the last inequality, we have used the fact that $\partial_x u$ is not identically zero.

(b) Use the condition on E to say what happens to $u(x, t)$ as $t \rightarrow \infty$. You might want to also think about this physically.

Notice that $E(t)$ is the integral of a non-negative function, so $E(t) \geq 0$ for all t . However, E is also a decreasing function, so we must have $E(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus

$$\int_{-\infty}^{\infty} (u(x, t))^2 dx \rightarrow 0,$$

which can only happen if $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for all x . Physically, this makes sense; all the heat should leak out to $\pm\infty$, and so temperature should fall to 0.