Practice Problems

Math 3150

October 30, 2003

1. Consider the wave equation
$$\partial_t^2 u = \partial_x^2 u, \qquad u(0,t) = 0 = u(1,t),$$
where the intial velocity $\partial_t u(x,0)$ is zero and the initial position is given by
$$u(x,0) = \begin{cases} 0 & 0 \le x < 1/4 \\ 1 & 1/4 \le x \le 3/4 \\ 0 & 3/4 < x \le 1. \end{cases}$$
What is the smallest positive time t such that $u(7/8,t) \neq 0$? (Hint: use d'Alembert's solution.)

2. Solve the following heat equations.

   (a) $\partial_t u = \partial_x^2 u$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = 2\sin(2x) - \sin(3x)$.

   (b) $\partial_t u = \partial_x^2 u$, $u(0,t) = 0 = u(1,t)$, $u(x,0) = 2\sin(4\pi x) - \sin(\pi x)$.

   (c) $\partial_t u = \partial_x^2 u$, $u(0,t) = 0 = u(1,t)$, $u(x,0) = x(1-x)$.

   (d) $\partial_t u = \partial_x^2 u$, $u(0,t) = 0 = u(1,t)$, $u(x,0) = 1/2 + |x - 1/2|$.

   (e) $\partial_t u = \partial_x^2 u$, $u(0,t) = 0$, $u(\pi,t) = \pi$, $u(x,0) = 2\sin(2x) - \sin(3x) + \pi x$.

   (f) $\partial_t u = \partial_x^2 u$, $u(0,t) = 2$,$u(1,t) = 0$, $u(x,0) = \sin(4\pi x) - \sin(\pi x) + 2 - 2x$.

   (g) $\partial_t u = \partial_x^2 u$, $\partial_x u(0,t) = 0 = \partial_x u(\pi,t)$, $u(x,0) = 3\sin(x) + 2\sin(4x)$.

   (h) $\partial_t u = \partial_x^2 u$, $\partial_x u(0,t) = 0 = \partial_x u(1,t)$, $u(x,0) = -\sin(2\pi x) + 2\sin(\pi x)$.

   (i) $\partial_t u = \partial_x^2 u$, $\partial_x u(0,t) = 0 = \partial_x u(1,t)$, $u(x,0) = 1/2 + |x - 1/2|$

3. (a) Consider the heat equation
$$\partial_t u = \partial_x^2 u, \qquad u(0,t) = 0 = u(L,t), \qquad u(x,0) = f(x)$$
and define the energy at time t as
$$E(t) := \int_0^L u^2(x,t)dx.$$
Show that the energy is a decreasing function in time. Does this make sense physically? (Hint: how can you characterize decreasing functions of one variable? You might want to use the equation and the boundary conditions, and integrate by parts at some point.)

   (b) Consider the same heat equation as in the previous part, but this time with the boundary conditions
$$\partial_x u(0,t) = 0 = \partial_x(L,t).$$
Is the energy still decreasing? Explain your answer.

4. Solve the following differential equations on the square $\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$.

   (a) $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$,
   $u(x,y,0) = \sin(\pi x)\sin(2\pi y)$, $\partial_t u(x,y,0) = 0$

   (b) $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$,
   $u(x,y,0) = \sin(\pi x)\sin(2\pi y)$, $\partial_t u(x,y,0) = \sin(3\pi x)\sin(2\pi y)$

   (c) $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$,
   $u(x,y,0) = \sin(\pi x)$, $\partial_t u(x,y,0) = 0$

   (d) $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$,
   $u(x,y,0) = xy(1-x)(1-y)$, $\partial_t u(x,y,0) = 0$

   (e) $\partial_t u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$, $u(x,y,0) = \sin(3\pi x)\sin(2\pi y)$

   (f) $\partial_t u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$, $u(x,y,0) = \sin(2\pi y)$

   (g) $\partial_t u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$, $u(x,y,0) = 1/2 + |x - 1/2|$

   (h) $\partial_t u = \partial_x^2 u + \partial_y^2 u$, $\partial_x u(0,y,t) = \partial_x u(1,y,t) = \partial_y u(x,0,t) = \partial_y u(x,1,t) = 0$, $u(x,y,0) = \cos(2\pi x)\cos(\pi y)$

(i) $\partial_t u = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = 0$, $u(1,y,t) = 1$, $u(x,0,t) = x = u(x,1,t)$, $u(x,y,0) = x + \sin(4\pi x)\sin(3\pi y)$

(j) $0 = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = u(1,y,t) = u(x,0,t) = 0$, $u(x,1,t) = 1$.

(k) $0 = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = 1$, $u(1,y,t) = 2$, $u(x,0,t) = 0 = u(x,1,t)$.

(l) $0 = \partial_x^2 u + \partial_y^2 u$, $u(0,y,t) = y = u(1,y,t)$, $u(x,0,t) = 0$, $u(x,1,t) = 1$. (Hint: you might be able to guess the solution to this one.)

5. Consider the differential equation

$$\partial_t u = \partial_x^2 u + \partial_y^2 u, \qquad u(x,y,0) = f(x,y)$$

with the boundary conditions

$$u(x,0,t) = 0 = u(x,1,t), \qquad \partial_x u(0,y,t) = 0 = \partial_x u(1,y,t).$$

(a) Using the form $u(x,y,t) = X(x)Y(y)T(t)$, separate variables and write down the differential equations X, Y and T must satisfy.

(b) What are the boundary conditions for X and Y?

(c) What are the initial conditions for T?

(d) Write down what X and Y are from the boundary conditions.

(e) What is the form of the general solution to this equation?

6. Consider the differential equation

$$\partial_t u + u = \partial_x^2 u, \qquad u(0,t) = 0 = u(L,t), \qquad u(x,0) = f(x).$$

Separate variables and write down the general solution to this equation.

7. For each of the following functions of two variables, compute the Laplacian $\Delta u = \partial_x^2 u + \partial_y^2 u$. You may want to use a different coordinate system for some of these.

(a) $u = x^2 - y^2$

(b) $u = x^3 - 3x^2 y + 3xy^2 - y^3$

(c) $u = (x^2 + y^2)^{-3/2}$

(d) $u = \arctan(y/x)$

(e) $u = (x^2 + y^2)\arctan(y/x)$

8. For each of the give functions $f = f(\theta)$ on the unit circle in the plane, solve the boundary value problem

$$\Delta u(r,\theta) = 0 \text{ for } r \leq 1, \qquad u(1,\theta) = f(\theta).$$

(a) $f(\theta) = \cos(\theta) + \sin(2\theta)$

(b) $f(\theta) = \cos^2\theta - \sin^2\theta$

(c) $f(\theta) = \begin{cases} \pi - \theta & 0 \leq \theta \leq \pi \\ 0 & \pi \leq \theta < 2\pi \end{cases}$