

Solutions to the Practice Problems  
Math 3150  
November 3, 2003

1. Consider the wave equation

$$\partial_t^2 u = \partial_x^2 u, \quad u(0, t) = 0 = u(1, t),$$

where the initial velocity  $\partial_t u(x, 0)$  is zero and the initial position is given by

$$u(x, 0) = \begin{cases} 0 & 0 \leq x < 1/4 \\ 1 & 1/4 \leq x \leq 3/4 \\ 0 & 3/4 < x \leq 1. \end{cases}$$

What is the smallest positive time  $t$  such that  $u(7/8, t) \neq 0$ ? (Hint: use d'Alembert's solution.)

In this case, d'Alembert's solution is given by

$$u(x, t) = \frac{1}{2}[f(x+t) + f(x-t)],$$

which is the sum of a left moving pulse (the  $f(x+t)$  term) and a right moving pulse (the  $f(x-t)$  term). Therefore,  $u(7/8, t)$  is nonzero as soon as the right edge of the right moving pulse hits  $x = 7/8$ . Since the pulse moves with speed 1 and the right edge of the initial pulse is  $x = 3/4$ , this happens when

$$t = 7/8 - 3/4 = 1/8.$$

2. Solve the following heat equations.

(a)  $\partial_t u = \partial_x^2 u$ ,  $u(0, t) = 0 = u(\pi, t)$ ,  $u(x, 0) = 2 \sin(2x) - \sin(3x)$ .

The general solution to this type of a heat equation is

$$u(x, t) = \sum a_n e^{-(cn\pi)^2 t/L^2} \sin((n\pi x)/L),$$

where  $a_n$  are the Fourier sine coefficients for the initial data  $f$ . In this case,  $f$  is already in Fourier series, so we can read off that

$$a_2 = 2, \quad a_3 = -1, \quad a_n = 0 \text{ for } n \neq 2, 3.$$

Also, we have  $L = \pi$  and  $c = 1$ . Putting this all together, we have

$$u(x, t) = 2e^{-4t} \sin(2x) - e^{-9t} \sin(3x).$$

(b)  $\partial_t u = \partial_x^2 u$ ,  $u(0, t) = 0 = u(1, t)$ ,  $u(x, 0) = 2 \sin(4\pi x) - \sin(\pi x)$ .

Same general form as before. This time,  $L = 1 = c$  and

$$a_1 = -1, \quad a_4 = 2, \quad a_n = 0 \text{ otherwise.}$$

So

$$u(x, t) = -e^{-\pi^2 t} \sin(\pi x) + 2e^{-16\pi^2 t} \sin(4\pi x).$$

(c)  $\partial_t u = \partial_x^2 u$ ,  $u(0, t) = 0 = u(1, t)$ ,  $u(x, 0) = x(1-x)$ .

This time we have to do some work to find the Fourier series of  $f$ . We have that

$$\begin{aligned} a_n &= 2 \int_0^1 f(x) \sin(n\pi x) dx = 2 \int_0^1 x(1-x) \sin(n\pi x) dx = 2 \int_0^1 x \sin(n\pi x) dx - 2 \int_0^1 x^2 \sin(n\pi x) dx \\ &= 2 \left[ -\frac{x}{n\pi} \cos(n\pi x) \Big|_0^1 + \frac{1}{n\pi} \int_0^1 \cos(n\pi x) dx \right] - 2 \left[ -\frac{x^2}{n\pi} \cos(n\pi x) \Big|_0^1 + \frac{2}{n\pi} \int_0^1 x \cos(n\pi x) dx \right] \\ &= \frac{4}{n\pi} \int_0^1 x \cos(n\pi x) dx = \frac{4}{n\pi} \left[ \frac{x}{n\pi} \sin(n\pi x) \Big|_0^1 - \frac{1}{n\pi} \int_0^1 \sin(n\pi x) dx \right] = -\frac{4}{n^2 \pi^2} \int_0^1 \sin(n\pi x) dx \\ &= \frac{4}{n^3 \pi^3} \cos(n\pi x) \Big|_0^1 = \frac{4}{n^3 \pi^3} ((-1)^n - 1). \end{aligned}$$

This last quantity is 0 if  $n$  is even and  $-8/(n^3 \pi^3)$  if  $n$  is odd. So the Fourier series for  $f$  is

$$f(x) = x(1-x) = -\frac{8}{\pi^3} \sum_{n \text{ odd}} \frac{1}{n^3} \sin(n\pi x).$$

Therefore, we have

$$u(x, t) = -\frac{8}{\pi^3} \sum_{n \text{ odd}} e^{-\pi^2 n^2 t} \sin(n\pi x).$$

(d)  $\partial_t u = \partial_x^2 u$ ,  $u(0, t) = 0 = u(1, t)$ ,  $u(x, 0) = 1/2 + |x - 1/2|$ .

In this problem, we actually want to work with  $f(x) = 1/2 - |x - 1/2|$ . Then the analysis below carries through.

Again, we need to do some work to find the Fourier series of  $f$ . This time it will help to use the symmetry of the function, namely that  $f(x) = f(1 - x)$ . Because  $\sin(n\pi x)$  has the same symmetry, this means

$$\int_0^1 f(x) \sin(n\pi x) dx = 2 \int_0^{1/2} f(x) \sin(n\pi x) dx.$$

The Fourier coefficient is given by

$$\begin{aligned} a_n &= 2 \int_0^1 f(x) \sin(n\pi x) dx = 4 \int_0^{1/2} f(x) \sin(n\pi x) dx = 4 \int_0^{1/2} x \sin(n\pi x) dx \\ &= \frac{4}{n\pi} [-x \cos(n\pi x)]_0^{1/2} + \int_0^{1/2} \cos(n\pi x) dx = \frac{4}{n\pi} [-\frac{1}{2} \cos(n\pi/2) + \frac{1}{n\pi} \sin(n\pi x)]_0^{1/2} \\ &= \frac{4}{n\pi} [\frac{1}{n\pi} \sin(n\pi/2) - \frac{1}{2} \cos(n\pi/2)]. \end{aligned}$$

Thus the solution to the heat equation is given by

$$u(x, t) = \frac{4}{\pi} \sum \left[ \frac{1}{n^2 \pi} \sin(n\pi/2) - \frac{1}{2n} \cos(n\pi/2) \right] e^{-n^2 \pi^2 t} \sin(n\pi x).$$

(e)  $\partial_t u = \partial_x^2 u$ ,  $u(0, t) = 0$ ,  $u(\pi, t) = \pi$ ,  $u(x, 0) = 2 \sin(2x) - \sin(3x) + \pi x$ .

This time the boundary data is not zero, but it is constant. So we need to subtract off the steady state solution  $u_0(x)$ , which is the linear function interpolating the two boundary data points. We have

$$u_0(x) = \pi x.$$

Then our modified heat equation is

$$\partial_t \tilde{u} = \partial_x^2 \tilde{u}, \quad \tilde{u}(0, t) = 0 = \tilde{u}(\pi, t), \quad \tilde{u}(x, 0) = f(x) - u_0(x) = 2 \sin(2x) - \sin(3x).$$

We can read off the Fourier coefficients for this last equation, and they are

$$a_2 = 2, \quad a_3 = -1, \quad a_n = 0 \text{ otherwise.}$$

Therefore, we must have

$$\tilde{u}(x, t) = 2e^{-4t} \sin(2x) - e^{-9t} \sin(3x).$$

Putting this all together, we have

$$u(x, t) = \tilde{u}(x, t) + u_0(x, t) = \pi x + 2e^{-4t} \sin(2x) - e^{-9t} \sin(3x).$$

(f)  $\partial_t u = \partial_x^2 u$ ,  $u(0, t) = 2$ ,  $u(1, t) = 0$ ,  $u(x, 0) = \sin(4\pi x) - \sin(\pi x) + 2 - 2x$ .

Again, we have to find the steady state solution  $u_0(x)$ , which is the linear function interpolating between the boundary points. We have

$$u_0(x) = 2 - 2x.$$

The modified problem is then

$$\partial_t \tilde{u} = \partial_x^2 \tilde{u}, \quad \tilde{u}(0, t) = 0 = \tilde{u}(1, t), \quad \tilde{u}(x, 0) = f(x) - u_0(x) = \sin(4\pi x) - \sin(\pi x).$$

We can read off the Fourier coefficients for this last equation, and they are

$$a_1 = -1, \quad a_4 = 1, \quad a_n = 0 \text{ otherwise.}$$

Therefore, we must have

$$\tilde{u}(x, t) = -e^{-\pi^2 t} \sin(\pi x) + e^{-16\pi^2 t} \sin(4\pi x).$$

Putting this all together, we have

$$u(x, t) = \tilde{u}(x, t) + u_0(x) = 2 - 2x - e^{-\pi^2 t} \sin(\pi x) + e^{-16\pi^2 t} \sin(4\pi x).$$

(g)  $\partial_t u = \partial_x^2 u$ ,  $\partial_x u(0, t) = 0 = \partial_x u(\pi, t)$ ,  $u(x, 0) = 3 \sin(x) + 2 \sin(4x)$ .

Replace sin with cos above. The general solution for a heat equation of this type is

$$u(x, t) = a_0 + \sum a_n e^{-n^2 t} \cos(nx)$$

where

$$u(x, 0) = a_0 + \sum a_n \cos(nx).$$

The initial data  $u(x, 0) = 3 \cos(x) + 2 \cos(4x)$  is already in Fourier series, so we can read off the coefficients  $a_n$ :

$$a_1 = 3, \quad a_4 = 2, \quad a_n = 0 \text{ otherwise.}$$

Thus the solution is

$$u(x, t) = 3e^{-t} \cos(x) + 2e^{-16t} \cos(4x).$$

(h)  $\partial_t u = \partial_x^2 u$ ,  $\partial_x u(0, t) = 0 = \partial_x u(1, t)$ ,  $u(x, 0) = -\sin(2\pi x) + 2 \sin(\pi x)$ .

Again, replace sin with cos above. This time the general solution is

$$u(x, t) = a_0 + \sum a_n e^{-n^2 \pi^2 t} \cos(n\pi x).$$

We can read off the Fourier coefficients  $a_n$ , and they are

$$a_1 = 2, \quad a_2 = -1, \quad a_n = 0 \text{ otherwise.}$$

Thus the solution is

$$u(x, t) = 2e^{-\pi^2 t} \cos(\pi x) - e^{-4\pi^2 t} \cos(2\pi x).$$

(i)  $\partial_t u = \partial_x^2 u$ ,  $\partial_x u(0, t) = 0 = \partial_x u(1, t)$ ,  $u(x, 0) = 1/2 + |x - 1/2|$

Again, the general solution has the form

$$u(x, t) = a_0 + \sum a_n e^{-n^2 \pi^2 t} \cos(n\pi x).$$

This time, we have to do some work to find the Fourier coefficients  $a_n$ . First, we have that

$$a_0 = \int_0^1 f(x) dx = 2 \int_0^{1/2} x dx = \frac{1}{4}.$$

3. (a) Consider the heat equation

$$\partial_t u = \partial_x^2 u, \quad u(0, t) = 0 = u(L, t), \quad u(x, 0) = f(x)$$

and define the energy at time  $t$  as

$$E(t) := \int_0^L u^2(x, t) dx.$$

Show that the energy is a decreasing function in time. Does this make sense physically? (Hint: how can you characterize decreasing functions of one variable? You might want to use the equation and the boundary conditions, and integrate by parts at some point.)

We compute the derivative of  $E$ :

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \int_0^L u^2(x, t) dx = \int_0^L \partial_t u^2(x, t) dx = 2 \int_0^L u(x, t) \partial_t u(x, t) dx \\ &= 2 \int_0^L u(x, t) \partial_x^2 u(x, t) dx = 2u(x, t) \partial_x u(x, t) \Big|_{x=0}^{x=L} - 2 \int_0^L (\partial_x u(x, t))^2 dx \\ &= -2 \int_0^L (\partial_x u(x, t))^2 dx \leq 0. \end{aligned}$$

Here we have used the equation for  $u$ , the boundary conditions  $\partial_x u(0, t) = 0 = \partial_x u(L, t)$ , and integration by parts.

This makes sense physically, because the magnitude of temperature distribution is always decreasing in time.

(b) Consider the same heat equation as in the previous part, but this time with the boundary conditions

$$\partial_x u(0, t) = 0 = \partial_x(L, t).$$

Is the energy still decreasing? Explain your answer.

Yes, the energy is still decreasing, because the boundary term in the integration by parts is still zero.

4. Solve the following differential equations on the square  $\{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ .

(a)  $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ,  
 $u(x, y, 0) = \sin(\pi x) \sin(2\pi y)$ ,  $\partial_t u(x, y, 0) = 0$

We have that the general solution is of the form

$$u(x, y, t) = \sum \sin(n\pi x) \sin(m\pi y) [a_{n,m} \cos(\pi\sqrt{n^2 + m^2}t) + b_{n,m} \sin(\pi\sqrt{m^2 + n^2}t)]$$

where

$$u(x, y, 0) = \sum a_{n,m} \sin(n\pi x) \sin(m\pi y), \quad \partial_t u(x, y, 0) = \sum \pi\sqrt{n^2 + m^2} b_{n,m} \sin(n\pi x) \sin(m\pi y).$$

Here the initial velocity is 0, so we have  $b_{n,m} = 0$  for all  $n$  and  $m$ . Also, the initial position is already in Fourier series, so

$$a_{1,2} = 1, \quad a_{n,m} = 0 \text{ otherwise.}$$

Thus the solution is

$$u(x, y, t) = \sin(\pi x) \sin(2\pi y) \cos(\sqrt{5}\pi t).$$

(b)  $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ,  
 $u(x, y, 0) = \sin(\pi x) \sin(2\pi y)$ ,  $\partial_t u(x, y, 0) = \sin(3\pi x) \sin(2\pi y)$

We have the same general solution as above. Both the initial position and the initial velocity are already in Fourier series, so we can read off the coefficients  $a_{n,m}$  and  $b_{n,m}$ :

$$a_{1,2} = 1, \quad a_{n,m} = 0 \text{ otherwise}$$

and

$$b_{3,2} = \frac{1}{\sqrt{13}}, \quad b_{n,m} = 0 \text{ otherwise.}$$

Thus the solution is

$$u(x, y, t) = \sin(\pi x) \sin(2\pi y) \cos(\sqrt{5}\pi t) + \frac{1}{\sqrt{13}} \sin(3\pi x) \sin(2\pi y) \sin(\sqrt{13}\pi t).$$

(c)  $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ,  
 $u(x, y, 0) = \sin(\pi x)$ ,  $\partial_t u(x, y, 0) = 0$

Because the initial velocity is zero, we have  $b_{n,m} = 0$  for all  $n$  and  $m$ . To compute the coefficients  $a_{n,m}$ , we have

$$\begin{aligned} a_{n,m} &= 4 \int_0^1 \int_0^1 f(x, y) \sin(n\pi x) \sin(m\pi y) dx dy = 4 \int_0^1 \int_0^1 \sin(\pi x) \sin(n\pi x) \sin(m\pi y) dx dy = \\ &= 4 \int_0^1 \sin(m\pi y) dy \cdot \int_0^1 \sin(\pi x) \sin(n\pi x) dx = \left[-\frac{2}{m\pi} \cos(m\pi y)\right]_0^1 \cdot 2 \int_0^1 \sin(\pi x) \sin(n\pi x) dx \\ &= \frac{2}{m\pi} (1 - (-1)^m) 2 \int_0^1 \sin(\pi x) \sin(n\pi x) dx \end{aligned}$$

This last quantity is  $4/(m\pi)$  for  $m$  odd and  $n = 1$ , and zero otherwise. We have used the orthogonality properties of the functions  $\sin(n\pi x)$  here. Thus the solution is

$$u(x, y, t) = \sum_{m \text{ odd}} \frac{4}{m\pi} \sin(\pi x) \sin(m\pi y) \cos(\pi\sqrt{1 + m^2}t).$$

- (d)  $\partial_t^2 u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ,  
 $u(x, y, 0) = xy(1-x)(1-y)$ ,  $\partial_t u(x, y, 0) = 0$

The general solution has the form

$$u(x, y, t) = \sum \sin(n\pi x) \sin(m\pi y) [a_{n,m} \cos(\pi\sqrt{n^2 + m^2}t) + b_{n,m} \sin(\pi\sqrt{n^2 + m^2}t)].$$

The initial velocity is zero, so  $b_{n,m} = 0$  for all  $n$  and  $m$ . However, we need to do a little work to find the Fourier coefficients  $a_{n,m}$ :

$$\begin{aligned} a_{n,m} &= 4 \int_0^1 \int_0^1 f(x, y) \sin(n\pi x) \sin(m\pi y) dx dy = 4 \int_0^1 \int_0^1 xy(1-x)(1-y) \sin(n\pi x) \sin(m\pi y) dx dy \\ &= (2 \int_0^1 x(1-x) \sin(n\pi x) dx) \cdot (2 \int_0^1 y(1-y) \sin(m\pi y) dy) = \left[ \frac{4}{n^3\pi^3} ((-1)^n - 1) \right] \cdot \left[ \frac{4}{m^3\pi^3} ((-1)^m - 1) \right] \end{aligned}$$

This last quantity is 0 if either  $n$  or  $m$  is even. If both  $m$  and  $n$  are odd, it is  $16/(n^3 m^3 \pi^6)$ . Thus the solution is

$$u(x, y) = \sum_{n,m \text{ odd}} \frac{16}{n^3 m^3 \pi^6} \sin(n\pi x) \sin(m\pi y) \cos(\pi\sqrt{n^2 + m^2}t).$$

- (e)  $\partial_t u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ,  $u(x, y, 0) = \sin(3\pi x) \sin(2\pi y)$   
 Here the general solution of this heat equation is

$$u(x, y, t) = \sum a_{n,m} e^{-(n^2+m^2)\pi^2 t} \sin(n\pi x) \sin(m\pi y)$$

where

$$u(x, y, 0) = \sum a_{n,m} \sin(n\pi x) \sin(m\pi y).$$

Here the initial temperature is already in Fourier series, so we have

$$a_{3,2} = 1, \quad a_{n,m} = 0 \text{ otherwise.}$$

Thus the solution is

$$u(x, y, t) = e^{-13\pi^2 t} \sin(3\pi x) \sin(2\pi y).$$

- (f)  $\partial_t u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ,  $u(x, y, 0) = \sin(2\pi y)$

This time we have to do a little work to find the Fourier coefficients. We have

$$\begin{aligned} a_{n,m} &= 4 \int_0^1 \int_0^1 \sin(2\pi y) \sin(n\pi x) \sin(m\pi y) dx dy = (2 \int_0^1 \sin(n\pi x) dx) \cdot (2 \int_0^1 \sin(2\pi y) \sin(m\pi y) dy) \\ &= \frac{2}{n\pi} (1 - (-1)^n) \cdot (2 \int_0^1 \sin(2\pi y) \sin(m\pi y) dy). \end{aligned}$$

This last quantity is  $4/(n\pi)$  for  $n$  odd and  $m = 2$  and 0 otherwise. Thus the solution is

$$u(x, y, t) = \sum_{n \text{ odd}} \frac{4}{n\pi} e^{-\pi^2(4+n^2)t} \sin(n\pi x) \sin(2\pi y).$$

- (g)  $\partial_t u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$ ,  $u(x, y, 0) = 1/2 + |x - 1/2|$

- (h)  $\partial_t u = \partial_x^2 u + \partial_y^2 u$ ,  $\partial_x u(0, y, t) = \partial_x u(1, y, t) = \partial_y u(x, 0, t) = \partial_y u(x, 1, t) = 0$ ,  $u(x, y, 0) = \cos(2\pi x) \cos(\pi y)$

This time the general solution is

$$u(x, y, t) = a_0 + \sum e^{-\pi^2(n^2+m^2)t} \cos(n\pi x) \cos(m\pi y)$$

where

$$u(x, y, 0) = a_0 + \sum a_{n,m} \cos(n\pi x) \cos(m\pi y).$$

Fortunately, our initial temperature is already in Fourier series, so we can read off the Fourier coefficients:

$$a_{2,1} = 1, \quad a_{n,m} = 0 \text{ otherwise.}$$

Thus we have

$$u(x, y, t) = e^{-5\pi^2 t} \cos(2\pi x) \cos(\pi y).$$

- (i)  $\partial_t u = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = 0$ ,  $u(1, y, t) = 1$ ,  $u(x, 0, t) = x = u(x, 1, t)$ ,  $u(x, y, 0) = x + \sin(4\pi x) \sin(3\pi y)$   
 This time we have to subtract off the steady state solution  $u_0(x, y)$ . It turns out that  $u_0 = x$  (see below), so our modified problem is

$$\partial_t u_1 = \partial_x^2 u_1 + \partial_y^2 u_1, \quad u_1(0, y, t) = u_1(1, y, t) = u_1(x, 0, t) = u_1(x, 1, t), \quad u_1(x, y, 0) = \sin(4\pi x) \sin(3\pi y).$$

The initial temperature is already in Fourier series, so we can read off that the Fourier coefficients are

$$a_{4,3} = 1, \quad a_{n,m} = 0 \text{ otherwise.}$$

Therefore,

$$u_1(x, y, t) = e^{-25\pi^2 t} \sin(4\pi x) \sin(3\pi y)$$

and the whole solution is

$$u(x, y, t) = u_0 + u_1 = x + e^{-25\pi^2 t} \sin(4\pi x) \sin(3\pi y).$$

- (j)  $0 = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = u(1, y, t) = u(x, 0, t) = 0$ ,  $u(x, 1, t) = 1$ .

We will separate variables. Let  $u(x, y) = X(x)Y(y)$ . Then the equation becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2,$$

with the boundary conditions

$$X(0) = 0 = X(1), \quad Y(0) = 0, \quad Y(1) = \text{constant.}$$

The boundary conditions on  $X$  force

$$X(x) = \sin(n\pi x), \quad \lambda = n\pi, \quad n = 1, 2, 3, \dots$$

Then the equation for  $Y$  becomes

$$Y'' = n^2 \pi^2 Y, \quad Y(0) = 0, \quad Y(1) = a_n.$$

Therefore we have  $Y(y) = a_n \sinh(n\pi y)$ , so

$$u(x, y) = \sum a_n \sin(n\pi x) \sinh(n\pi y),$$

where

$$1 = \sum a_n \sinh(n\pi) \sin(n\pi x).$$

It remains to find the Fourier series for the function 1. These Fourier coefficients are

$$c_n = 2 \int_0^1 \sin(n\pi x) dx = -\frac{2}{n\pi} \cos(n\pi x) \Big|_0^1 = \frac{2}{n\pi} (1 - (-1)^n).$$

This last quantity is 0 for  $n$  even and  $4/(n\pi)$  for  $n$  odd. Thus we have

$$a_n \sinh(n\pi) = \frac{2}{n\pi} (1 - (-1)^n),$$

so

$$u(x, y) = \sum_{n \text{ odd}} \frac{4 \sinh(n\pi)}{n\pi} \sinh(n\pi y) \sin(n\pi x).$$

- (k)  $0 = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = 1$ ,  $u(1, y, t) = 2$ ,  $u(x, 0, t) = 0 = u(x, 1, t)$ .

Again, we separate variables and write  $u(x, y) = X(x)Y(y)$ . Then

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda^2,$$

with the boundary conditions

$$X(0) = 1, \quad X(1) = 2, \quad Y(0) = 0 = Y(1).$$

Solving for the equation in  $Y$  first, we see

$$Y(y) = \sin(n\pi y), \quad \lambda = n\pi.$$

Thus the equation for  $X$  is  $X'' = n^2\pi^2 X$  and

$$X(x) = a_n \cosh(n\pi x) + b_n \sinh(n\pi x).$$

Our solution is

$$u(x, y) = \sum \sin(n\pi y)[a_n \cosh(n\pi x) + b_n \sinh(n\pi x)].$$

We can determine the coefficients by matching the boundary data. Evaluating at  $x = 0$  we have

$$1 = u(0, y) = \sum a_n \sin(n\pi y) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(n\pi y),$$

so  $a_n = 0$  for  $n$  even and  $a_n = 4/(n\pi)$  for  $n$  odd. Evaluating at  $x = 1$ , we have

$$2 = u(1, y) = \sum \sin(n\pi y)[a_n \cosh(n\pi) + b_n \sinh(n\pi)] = \sum_{n \text{ odd}} \frac{8}{n\pi} \sin(n\pi y).$$

So we have  $b_n = 0$  for  $n$  odd and, for  $n$  even, we have

$$b_n = \frac{4}{n\pi} \cdot \frac{2 - \cosh(n\pi)}{\sinh(n\pi)}.$$

Thus we have

$$u(x, y) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(n\pi y) \left[ \cosh(n\pi x) + \frac{(2 - \cosh(n\pi))}{\sinh(n\pi)} \sinh(n\pi x) \right].$$

- (1)  $0 = \partial_x^2 u + \partial_y^2 u$ ,  $u(0, y, t) = y = u(1, y, t)$ ,  $u(x, 0, t) = 0$ ,  $u(x, 1, t) = 1$ . (Hint: you might be able to guess the solution to this one.)

This is one of those problems where it helps to draw a picture. The boundary data for this problem all lies in the plane  $z = x$ , which is the graph of the linear function  $u(x, y) = x$ . So the solution is just

$$u(x, y) = x.$$

5. Consider the differential equation

$$\partial_t u = \partial_x^2 u + \partial_y^2 u, \quad u(x, y, 0) = f(x, y)$$

with the boundary conditions

$$u(x, 0, t) = 0 = u(x, 1, t), \quad \partial_x u(0, y, t) = 0 = \partial_x u(1, y, t).$$

- (a) Using the form  $u(x, y, t) = X(x)Y(y)T(t)$ , separate variables and write down the differential equations  $X$ ,  $Y$  and  $T$  must satisfy.

Let  $u(x, y, t) = X(x)y(y)T(t)$ . Then the differential equation reads

$$XYT' = X''YT + XY''T,$$

which we can rearrange to read

$$\frac{T'}{T}(t) = \frac{X''}{X}(x) + \frac{Y''}{Y}(y).$$

Since one side of the equation depends on  $t$  alone, and the other side depends on  $x$  and  $y$ , both sides of the equation must be equal to a constant, which we will call  $-\lambda^2$ . Thus we have

$$\frac{T'}{T} = -\lambda^2, \quad \frac{X'}{X} + \frac{Y'}{Y} = -\lambda^2.$$

We need to separate variables again in the second of these two equations, which we can rearrange to read

$$\frac{X''}{X}(x) = -\lambda^2 - \frac{Y''}{Y}(y).$$

This time, one side of the equation depends on  $x$  alone while the other side depends on  $y$  alone, so both sides of the equation must be equal to a constant which we will call  $-\mu^2$ . The two equations we have now are

$$\frac{X''}{X} = -\mu^2, \quad \frac{Y''}{Y} = -(\lambda^2 - \mu^2) = -\nu^2.$$

The solution to these last two equations are

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x), \quad Y(y) = c_1^* \cos(\nu y) + c_2^* \sin(\nu y).$$

We also have

$$T(t) = c_3 e^{-\lambda^2 t}.$$

Here the relation between the separation constants  $\lambda$ ,  $\mu$  and  $\nu$  is given by

$$\lambda^2 = \mu^2 + \nu^2,$$

so the formula for  $T$  should really read

$$T(t) = c_3 e^{-(\mu^2 + \nu^2)t}.$$

- (b) What are the boundary conditions for  $X$  and  $Y$ ?

The boundary conditions on the equation for  $x$  come from the boundary conditions on the vertical sides, where we have  $\partial_x u(0, y, t) = 0 = \partial_x u(1, y, t)$ . Thus we have

$$X'(0) = 0 = X'(1).$$

Similarly, the boundary conditions on the equation for  $y$  come from the boundary conditions on the horizontal sides, where we have  $u(x, 0, t) = 0 = u(x, 1, t)$ . Thus we have

$$Y(0) = 0 = Y(1).$$

- (c) What are the initial conditions for  $T$ ?

The initial condition for  $T$  in finding  $T(0)$  is given by

$$f(x, y) = u(x, y, 0) = X(x)Y(y)T(0).$$

We'll return to this later.

- (d) Write down what  $X$  and  $Y$  are from the boundary conditions.

From these boundary condition at  $x = 0$  and the form of  $X$  above, we see that

$$X(x) = c_1 \cos(\mu x).$$

From the boundary condition at  $x = 1$  we have  $\mu = n\pi$  for  $n = 0, 1, 2, 3, \dots$ . Thus we may as well take

$$X(x) = \cos(n\pi x), \quad n = 0, 1, 2, 3, \dots$$

From the boundary condition at  $y = 0$  and the form of  $Y$  above, we see that

$$Y(y) = c_2^* \sin(\nu y).$$

From the boundary condition at  $y = 1$  we have  $\nu = m\pi$  for  $m = 0, 1, 2, 3, \dots$ . Thus we may as well take

$$Y(y) = \sin(m\pi y), \quad m = 0, 1, 2, 3, \dots$$

- (e) What is the form of the general solution to this equation?

So far we have

$$X(x) = \cos(n\pi x), \quad Y(y) = \sin(m\pi y), \quad T(t) = c_{n,m} e^{-(n^2 + m^2)t}.$$

To find the general solution we have to sum over all  $n = 0, 1, 2, 3, \dots$  and  $m = 0, 1, 2, 3, \dots$ . Thus the general form is

$$u(x, y, t) = a_0 + \sum_{n,m=1}^{\infty} a_{n,m} e^{-(n^2 + m^2)t} \cos(n\pi x) \sin(m\pi y),$$

where the coefficients  $a_{n,m}$  are given by

$$a_0 = \int_0^1 \int_0^1 f(x, y) dx dy, \quad a_{n,m} = 4 \int_0^1 \int_0^1 f(x, y) \cos(n\pi x) \sin(m\pi y) dx dy.$$

Alternatively, we could say

$$f(x, y) = a_0 + \sum_{n,m} a_{n,m} \cos(n\pi x) \sin(m\pi y).$$

6. Consider the differential equation

$$\partial_t u + u = \partial_x^2 u, \quad u(0, t) = 0 = u(L, t), \quad u(x, 0) = f(x).$$

Separate variables and write down the general solution to this equation.

We will try a solution of the form  $u(x, t) = X(x)T(t)$ . Then the equation reads

$$XT' + XT = X''T,$$

which we can rearrange to read

$$\frac{T' + T}{T} = \frac{X''}{X}.$$

The left hand side is a function of  $t$  alone, while the right hand side is a function of  $x$  alone, so both sides of the equation must be equal to a constant which we will call  $-\lambda^2$ . Now the equations are

$$\frac{T' + T}{T} = -\lambda^2, \quad \frac{X''}{X} = -\lambda^2.$$

Let's look at the equation for  $x$  first. We have the boundary conditions  $u(0, t) = 0 = u(L, t)$ , so

$$X(0) = 0 = X(L).$$

On the other hand, the general solution to the ODE for  $X$  is

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

Plugging in the boundary condition  $X(0) = 0$  we see  $c_1 = 0$ . Also,

$$0 = X(L) = c_2 \sin(\lambda L),$$

which implies  $\lambda L = n\pi$  for some  $n = 1, 2, 3, \dots$ , or

$$\lambda = \frac{n\pi}{L}.$$

We may as well take

$$X(x) = \sin((n\pi x)/L).$$

The differential equation for  $T$  is now

$$\frac{T' + T}{T} = -\frac{n^2 \pi^2}{L^2},$$

which we can rearrange to read

$$T' = -((n\pi^2/L^2 + 1)T).$$

The general solution to this equation is

$$T(t) = c_n e^{-(n^2 \pi^2/L^2 + 1)t}.$$

Putting this all together, we get that the general solution has the form

$$u(x, t) = \sum_1^{\infty} c_n e^{-(n^2 \pi^2/L^2 + 1)t} \sin((n\pi x)/L),$$

where  $c_n$  are the Fourier coefficients of the initial data  $f(x) = u(x, 0)$ .

7. For each of the following functions of two variables, compute the Laplacian  $\Delta u = \partial_x^2 u + \partial_y^2 u$ . You may want to use a different coordinate system for some of these.

(a)  $u = x^2 - y^2$

$$\Delta u = \partial_x^2(x^2 - y^2) + \partial_y^2(x^2 - y^2) = 2 - 2 = 0.$$

(b)  $u = x^3 - 3x^2y + 3xy^2 - y^3$

$$\Delta u = \partial_x^2(x^3 - 3x^2y + 3xy^2 - y^3) + \partial_y^2(x^3 - 3x^2y + 3xy^2 - y^3) = 6x + 6y + 6x - 6y = 12x$$

(c)  $u = (x^2 + y^2)^{-3/2}$

This time it's a good idea to use polar coordinates. Recall that

$$\Delta u = \partial_x^2 u + \partial_y^2 u = \partial_r^2 u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_\theta^2 u.$$

Our function is just  $u = r^{-3/2}$ , so

$$\Delta u = \partial_r^2(r^{-3/2}) + \frac{1}{r} \partial_r(r^{-3/2}) = \frac{9}{4} r^{-7/2} = \frac{9}{4} (x^2 + y^2)^{-7/2}.$$

(d)  $u = \arctan(y/x)$

Again, we should use polar coordinates to see that  $u(r, \theta) = \theta$ . Thus  $\Delta u = 0$ .

(e)  $u = (x^2 + y^2) \arctan(y/x)$

This time  $U = r^2 \theta$ , so

$$\Delta u = \partial_r^2(r^2 \theta) + \frac{1}{r} \partial_r(r^2 \theta) + \frac{1}{r^2} \partial_\theta^2(r^2 \theta) = \theta [\partial_r^2(r^2) + \frac{1}{r} \partial_r(r^2)] = 4\theta = 4 \arctan(y/x).$$

8. For each of the give functions  $f = f(\theta)$  on the unit circle in the plane, solve the boundary value problem

$$\Delta u(r, \theta) = 0 \text{ for } r \leq 1, \quad u(1, \theta) = f(\theta).$$

The general form of a harmonic function on the unit disc is

$$u(r, \theta) = a_0 + \sum r^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

where the boundary data is given by

$$u(1, \theta) = f(\theta) = a_0 + \sum [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

(a)  $f(\theta) = \cos(\theta) + \sin(2\theta)$

Here the boundary is already in Fourier series, so we can read off the coefficients:

$$a_1 = 1, \quad b_2 = 1,$$

and all the other coefficients are zero. Thus we have

$$u(r, \theta) = r \cos(\theta) + r^2 \sin(2\theta).$$

(b)  $f(\theta) = \cos^2 \theta - \sin^2 \theta$

This time we have to use a trig identity to get the boundary data in Fourier series. We have

$$f(\theta) = \cos^2 \theta - \sin^2 \theta = \cos(2\theta),$$

so  $a_2 = 1$  and all the other coefficients are zero. Therefore,

$$u(r, \theta) = r^2 \cos(2\theta).$$

(c)  $f(\theta) = \begin{cases} \pi - \theta & 0 \leq \theta \leq \pi \\ 0 & \pi \leq \theta < 2\pi \end{cases}$

This time we have to do some work to find the Fourier series for  $f$ . First we have

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = \frac{1}{2\pi} \int_0^\pi (\pi - \theta) d\theta = \frac{\pi}{2} - \frac{1}{2} \theta^2 \Big|_0^\pi = \frac{\pi}{4}.$$

For the  $a_n$  terms, we have

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta = \frac{1}{\pi} \int_0^\pi (\pi - \theta) \cos(n\theta) d\theta \\
 &= \int_0^\pi \cos(n\theta) d\theta - \frac{1}{\pi} \int_0^\pi \theta \cos(n\theta) d\theta = -\frac{1}{\pi} \left[ \frac{\theta}{n} \sin(n\theta) \right]_0^\pi - \frac{1}{n} \int_0^\pi \sin(n\theta) d\theta \\
 &= \frac{1}{n\pi} \int_0^\pi \int_0^\pi \sin(n\theta) d\theta = -\frac{1}{n^2\pi} \cos(n\theta) \Big|_0^\pi = \frac{1}{n^2\pi} [1 - (-1)^n].
 \end{aligned}$$

This last quantity is  $o$  for  $n$  even and  $2/(n^2\pi)$  for  $n$  odd. Finally, we have

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta = \int_0^\pi \sin(n\theta) - \frac{1}{\pi} \int_0^\pi \theta \sin(n\theta) d\theta \\
 &= -\frac{\cos(n\theta)}{n} \Big|_0^\pi - \frac{1}{\pi} \left[ -\frac{\theta}{n} \cos(n\theta) \right]_0^\pi + \frac{1}{n} \int_0^\pi \cos(n\theta) d\theta = \frac{1}{n} - \frac{(-1)^n}{n} - \frac{1}{\pi} \left( -\frac{\pi(-1)^n}{n} \right) = \frac{1}{n}.
 \end{aligned}$$

Thus we can write the solution as

$$u(r, \theta) = \frac{\pi}{4} + \sum_n r^n \left[ \frac{1}{\pi n^2} (1 - (-1)^n) \cos(n\theta) + \frac{1}{n} \sin(n\theta) \right].$$