Practice Problems

Math 3150

Sept. 22, 2003

1. Find the Fourier series for the following functions.

   (a) $f(x) = \cos(2x)$ for $-\pi \leq x \leq \pi$

   (b) $f(x) = x$ for $-2 \leq x \leq 2$.

   (c) $f(x) = x^2$ for $-1 \leq x \leq 1$.

   (d) $f(x) = |x|$ for $-\pi \leq x \leq \pi$.

   (e) $f(x) = \cos^2 x$ for $-\pi \leq x \leq x$ (Hint: you don't need to do any integrals!)

2. For each of the function on $[0, \pi]$ find both the even Fourier series (i.e. the Fourier cos series) and the odd Fourier series (i.e. the Fourier sin series).

   (a) $f(x) = \sin x$

   (b) $f(x) = x$

   (c) $f(x) = x^2$

3. Let $f(x) = e^{x^2}$ for $-\pi \leq x \leq \pi$

   (a) Is f an even function, an odd function, or neither?

   (b) **Without computing them,** what can you say about the Fourier coefficients b_k in the Fourier series for f?

4. Let f be an odd function on $[-\pi, \pi]$.

   (a) What is the average value of f?

   (b) What can you say about the Fourier coefficients of f?

   (c) Suppose that in addition f is π-periodic. Can you say anything more about the Fourier series of f?

5. Given the Fourier series

$$x^3 = \sum_1^{\infty} [\frac{12 - 2\pi^2 k^2}{k^3} (-1)^k \sin(kx)]$$

   for $\pi \leq x \leq \pi$, find the Fourier series for $g(x) = 1 - x^3/8$ on the same interval. (Hint: you don't need to do any integrals).

6. Given the Fourier series

$$x^2 = \frac{p^2}{3} + \frac{4p^2}{\pi} \sum_1^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi x/p), \qquad -p \leq x \leq p$$

   show that

$$\frac{\pi}{4} = \sum_1^{\infty} \frac{1}{k^2}.$$

7. Let f be a 2π-periodic function, defined one the whole real line. In addition, suppose f is smooth, i.e. f has infinitely many derivatives.

   (a) Is f' periodic? Justify your answer.

   (b) Let $F(x) = \int_0^x f(t)dt$. Is F periodic? Justify your answer.

8. Suppose f is π-periodic, g is 2π-periodic and h is 2-periodic. For each of the questions below, either show the given function is periodic (and find its period) or produce a counter example to show that it need not be periodic.

   (a) Is $f + g$ periodic?
   (b) Is $g + h$ periodic?
   (c) Is $g \circ h$ periodic?
   (d) Let $F(x) = f(x) + h(\pi x)$. Is F periodic?

9. Let f be a non-constant, smooth function on $[0, 1]$ such that

$$f'' = \lambda f, \qquad f(0) = 0 = f(1).$$

   Show that $\lambda < 0$. (Hint: integrate $\int_0^1 f(x) \cdot f''(x) dx$ by parts.)

10. For which of the PDEs listed below does the principle of superposition hold (i.e. when can you add solutions to obtain another solution)?

   (a) $u_{tt} = u_{xt}$
   (b) $u_{tt} = u_x - u_t$
   (c) $u_t^2 = u_{xx}$
   (d) $u_{tx} = u_{xx} - u_{tt}$

11. For each of the initial conditions listed below, solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

   for $0 \leq x \leq \pi$, $t \geq 0$ and the boundary conditions

$$u(0, t) = 0u(\pi, t)$$

   for all t.

   (a) $u(x, 0) = f(x) = \sin(2x), \partial_t u(x, 0) = g(x) = 0$
   (b) $u(x, 0) = f(x) = \sin x, \partial_t u(x, 0) = g(x) = \cos(3x)$
   (c) $u(x, 0) = f(x) = x(x - \pi), \partial_t u(x, 0) = g(x) = 0$
   (d) $u(x, 0) = f(x) = 0, \partial_t u(x, 0) = \frac{\tan x}{1+\tan^2 x}$ (Hint: you don't need to integrate anything to compute the Fourier series for g!)

12. The equation of motion for a damped vibrating string is

$$\partial_t^2 u + k\partial_t u = c^2 \partial_x^2 u.$$

   As in the case of a free vibrating string, we will consider the case where the endpoints of the string are fixed at the same height, the length of the string is L, and look for a solution of the form

$$u(x, t) = v(x)w(t).$$

   (a) What are the ODEs which v and w satisfy?
   (b) What boundary conditions and/or initial conditions must v and w satisfy?
   (c) Can you use these boundary/initial conditions to say anything about the solutions to the ODEs?