Solutions to the Practice Problems

Math 3150

Sept. 28, 2003

1. Find the Fourier series for the following functions.

   (a) $f(x) = \cos(2x)$ for $-\pi \le x \le \pi$
   THis function is already in Fourier series, so we don't need to do anything.

   (b) $f(x) = x$ for $-2 \le x \le 2$.
   First note that f is odd. Thus $a_0 = 0$ and $a_k = 0$ for $k = 1, 2, 3, \ldots$. As for the rest of the coefficients:

$$\begin{aligned} b_k &= \frac{1}{2}\int_{-2}^{2} x\sin(k\pi x/2)dx = \int_0^2 x\sin(k\pi x/2)dx = -\frac{2}{k\pi}\, x\cos(k\pi x/2)|_0^2 + \frac{2}{k\pi}\int_0^2 \cos(k\pi x/2)dx \\ &= -\frac{2\cos(k\pi)}{k\pi} = \frac{2(-1)^{k+1}}{k\pi}. \end{aligned}$$

   Thus the Fourier series is

$$f(x) = \frac{2}{\pi}\sum_1^\infty \frac{(-1)^{k+1}}{k}\sin(k\pi x/2).$$

   (c) $f(x) = x^2$ for $-1 \le x \le 1$.
   This time, f is even. So $b_k = 0$ for all k. First we compute a_0:

$$a_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \int_0^1 x^2 dx = \frac{1}{3}.$$

   Next we'll compute a_k for $k \ge 1$. We'll have to integrate by parts twice.

$$\begin{aligned} a_k &= \frac{1}{2}\int -1^1 x^2\cos(k\pi x)dx = \int_0^1 x^2\cos(k\pi x)dx \\ &= \frac{x^2}{k\pi}\sin(k\pi x)\Big|_0^1 - \frac{2}{k\pi}\int_0^1 x\sin(k\pi x)dx = -\frac{2}{k\pi}[-\frac{x}{k\pi}\cos(k\pi x)\Big|_0^1 + \frac{1}{k\pi}\int_0^1 \cos(k\pi x)dx] \\ &= \frac{2\cos(k\pi)}{k^2\pi^2} = \frac{2(-1)^k}{k^2\pi^2}. \end{aligned}$$

   Thus the Fourier series is

$$f(x) = \frac{1}{3} + \frac{2}{\pi^2}\sum_1^\infty \frac{(-1)^k}{k^2}\cos(k\pi x).$$

   (d) $f(x) = |x|$ for $-\pi \le x \le \pi$.
   Notice that f is even, so $b_k = 0$ for all k. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^\pi x dx = \frac{x^2}{2\pi}\Big|_0^\pi = \frac{\pi}{2}.$$

   Next we compute a_k:

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{2}{\pi}\int_0^\pi x\cos(kx)dx = \frac{2}{\pi}[\frac{1}{k}\, x\sin(kx)|_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)dx] \\ &= -\frac{2}{\pi k}\int_0^\pi \sin(kx)dx = \frac{2}{\pi k^2}\cos(kx)|_0^\pi = \frac{2}{\pi k^2}((-1)^k - 1). \end{aligned}$$

   This last quantity is 0 if k is even and $-4/(\pi k^2)$ if k is odd. Thus the Fourier series is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k \text{ odd}} \frac{1}{k^2}\cos(kx).$$

   (e) $f(x) = \cos^2 x$ for $-\pi \le x \le x$ (Hint: you don't need to do any integrals!)
   Here we use the trig identity $\cos^2 x = (1/2)(1 + \cos(2x))$. This function is already written as a Fourier series, so we're done.

2. For each of the function on $[0,\pi]$ find both the even Fourier series (i.e. the Fourier cos series) and the odd Fourier series (i.e. the Fourier sin series).

   (a) $f(x) = \sin x$

   This function is already in Fourier sine series, so we don't have to compute that.

   To compute the Fourier cosine series, we will use the identity

$$\sin x \cos(kx) = \frac{1}{2}[\sin((k+1)x) - \sin((k-1)x)].$$

   Then for $k = 1, 2, 3, \ldots$

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^\pi \sin x \cos(kx)dx = \frac{1}{\pi}\int_0^\pi (\sin((k+1)x) - \sin((k-1)x))dx \\ &= \frac{2}{\pi}[-\frac{1}{k+1}\cos((k+1)x)|_0^\pi + \frac{1}{k-1}\cos((k-1)x)|_0^\pi] \\ &= \frac{2}{\pi}[\frac{1}{k+1}(1-(-1)^{k+1}) - \frac{1}{k-1}(1-(-1)^{k-1})] \\ &= \frac{4}{\pi(k^2-1)}[1-(-1)^{k+1}] = \frac{4(1+(-1)^k)}{\pi(k^2-1)}. \end{aligned}$$

   This last quantity is 0 for k odd and $8/(\pi(k^2-1))$ for k even. Finally we compute

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin x dx = -\frac{1}{\pi}\cos x|_0^\pi = \frac{2}{\pi}.$$

   Thus we have

$$f(x) = \frac{2}{\pi} + \frac{8}{\pi}\sum_{k \text{ even}} \frac{1}{k^2-1}\cos(kx).$$

   (b) $f(x) = x$

   For the sine series:

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^\pi x\sin(kx)dx = \frac{2}{\pi}[-\frac{x}{k}\cos(kx)\Big|_0^\pi + \frac{1}{k}\int_0^\pi \cos(kx)dx] \\ &= -\frac{2\cos(k\pi)}{k} = \frac{2(-1)^{k+1}}{k}. \end{aligned}$$

   Thus the sine series is

$$x = \sum_1^\infty \frac{2(-1)^{k+1}}{k}\sin(kx).$$

   For the cosine series:

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^\pi x\cos(kx)dx = \frac{2}{\pi}[\frac{x}{k}\sin(kx)\Big|_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)dx] \\ &= \frac{2}{\pi k^2}\cos(kx)|_0^\pi = \frac{2}{\pi k^2}((-1)^k - 1). \end{aligned}$$

   This last quantity is 0 if k is even and $-4/(\pi k^2)$ if k is odd. Finally,

$$a_0 = \int_0^\pi x dx = \frac{\pi^2}{2}.$$

   Thus we have

$$x = \frac{\pi^2}{2} - \frac{4}{\pi}\sum_{k \text{ odd}} \frac{1}{k^2}\cos(kx).$$

   (c) $f(x) = x^2$

   This time we will have to integrate by parts twice.

For the sine series:

$$\begin{aligned} b_k &= \frac{2}{\pi}\int_0^\pi x^2\sin(kx)dx = \frac{2}{\pi}[-\frac{x^2}{k}\cos(kx)\Big|_0^\pi + \frac{2}{k}\int_0^\pi x\cos(kx)dx] \\ &= \frac{2}{\pi}[\frac{\pi^2(-1)^{k+1}}{k} + \frac{2}{k}(\frac{x}{k}\sin(kx)\Big|_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)dx)] = \frac{2\pi(-1)^{k+1}}{k} + \frac{2}{\pi k^3}\cos(kx)|_0^\pi \\ &= \frac{2\pi(-1)^{k+1}}{k} + \frac{2((-1)^k-1)}{k^3\pi}. \end{aligned}$$

For the cosine series:

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^\pi x^2\cos(kx)dx = \frac{2}{\pi}[\frac{x^2}{k}\sin(kx)\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi x\sin(kx)dx] = -\frac{4}{k\pi}\int_0^\pi x\sin(kx)dx \\ &= -\frac{4}{k\pi}[-\frac{x}{k}\cos(kx)\Big|_0^\pi + \frac{1}{k}\int_0^\pi \cos(kx)dx] = \frac{4}{k^2}(-1)^k. \end{aligned}$$

Also,

$$a_0 = \frac{1}{\pi}\int_0^\pi x^2dx = \frac{\pi^2}{3}.$$

3. Let $f(x) = e^{x^2}$ for $-\pi \le x \le \pi$

   (a) Is f an even function, an odd function, or neither?
   f is even: $f(-x) = e^{(-x)^2} = e^{x^2} = f(x)$.

   (b) **Without computing them,** what can you say about the Fourier coefficients b_k in the Fourier series for f?
   We know that $b_k = 0$. Indeed, these Fourier coefficients are given by the integral $\int_{-\pi}^{\pi} f(x)\sin(kx)dx$. This is the integral of an odd function over an interval centered about 0, so it must be zero.

4. Let f be an odd function on $[-\pi, \pi]$.

   (a) What is the average value of f?
   The average value of f is $(1/(2\pi))\int_{-\pi}^{\pi} f(x)dx = 0$, because f is odd.

   (b) What can you say about the Fourier coefficients of f?
   This time, the Fourier coefficients a_k are all zero.

   (c) Suppose that in addition f is π-periodic. Can you say anything more about the Fourier series of f?
   If, in addition, f is π periodic, then $b_1 = 0$. There are probably numerous ways to prove this, and here's one sketch of a proof. First note that

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x dx = \frac{1}{\pi}\int_0^\pi f(x)\sin x dx.$$

   Also, $\sin(x) > 0$ for $0 < x < \pi$. Meanwhile,

$$\int_0^\pi f(x)dx = \int_{-\pi/2}^{\pi} /2f(x)dx = 0.$$

   Putting this together with $\sin x > 0$ on $(0, \pi)$, we see that

$$\int_0^\pi f(x)\sin x dx = 0,$$

   and so $b_1 = 0$.

5. Given the Fourier series

$$x^3 = \sum_1^\infty [\frac{12 - 2\pi^2k^2}{k^3}(-1)^k\sin(kx)]$$

   for $\pi \le x \le \pi$, find the Fourier series for $g(x) = 1 - x^3/8$ on the same interval. (Hint: you don't need to do any integrals).

   All we need to do is substitute the given Fourier series for x^3:

$$1 - x^3/8 = 1 - \frac{1}{8}\sum_1^\infty [\frac{12 - 2\pi^2k^2}{k^3}(-1)^k\sin(kx)].$$

6. Given the Fourier series

$$x^2 = \frac{p^2}{3} + \frac{4p^2}{\pi}\sum_1^\infty \frac{(-1)^k}{k^2}\cos(k\pi x/p), \qquad -p \le x \le p$$

show that

$$\frac{\pi}{4} = \sum_1^\infty \frac{1}{k^2}.$$

I made a mistake in this problem: it should really read either $\pi/12 = \sum(-1)^{k+1}/k^2$ or $\pi/6 = \sum 1/k^2$. We'll compute both sums below.

For $\pi/12$, evaluate the function at $x = p/2$. Then we see

$$\frac{p^2}{4} = \frac{p^2}{3} + \frac{4p^2}{\pi}\sum_1^\infty \frac{(-1)^k}{k^2}\cos(k\pi/2) = \frac{p^2}{3} + \frac{4p^2}{\pi}\sum_{k \text{ even}} \frac{1}{k^2}(-1)^{k/2}.$$

Rearranging this equation and replacing k with $2l$, we find

$$-\frac{\pi}{12} = 4\sum_1^\infty \frac{(-1)^l}{(2l)^2} = \sum_1^\infty \frac{(-1)^l}{l^2},$$

which is the correct sum.

For $\pi/6$, evaluate the function at p. This time we have

$$p^2 = \frac{p^2}{3} + \frac{4p^2}{\pi}\sum_1^\infty \frac{(-1)^k}{k^2}\cos(k\pi) = \frac{p^2}{3} + \frac{4p^2}{\pi}\sum_1^\infty \frac{1}{k^2}.$$

Rearranging this last equation we find

$$\frac{\pi}{6} = \sum_1^\infty \frac{1}{k^2}.$$

7. Let f be a 2π-periodic function, defined one the whole real line. In addition, suppose f is smooth, i.e. f has infinitely many derivatives.

   (a) Is f' periodic? Justify your answer.

   Yes. One way to see this is to recall that a deriviative is a limit of difference quotients. These difference quotients are periodic because f is. Therefore, the limit of the difference quotients is the limit of periodic functions, so it too must be periodic. More precisely, the derivative at x_0 is

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x+2\pi)-f(x_0+2\pi)}{x-x_0} = \lim_{x\to x_0} \frac{f(x+2\pi)-f(x_0+2\pi)}{(x+2\pi)-(x_0+2\pi)} = f'(x_0+2\pi).$$

   (b) Let $F(x) = \int_0^x f(t)dt$. Is F periodic? Justify your answer.

   No. Consider $f(x) = \cos(x) + 1$. Then $F(x) = \sin(x) + x$, which is not periodic.

8. Suppose f is π-periodic, g is 2π-periodic and h is 2-periodic. For each of the questions below, either show the given function is periodic (and find its period) or produce a counter example to show that it need not be periodic.

   (a) Is $f + g$ periodic?

   Yes: $(f+g)(x+2\pi) = f(x+2\pi) + g(x+2\pi) = f(x) + g(x) = (f+g)(x)$.

   (b) Is $g + h$ periodic?

   No. Take (for example) $g(x) = \cos(x)$ and $h(x) = \cos(\pi x)$. We saw in the homework that this is not a periodic function.

   (c) Is $g \circ h$ periodic?

   Yes: $g \circ h(x+2) = g(h(x+2)) = g(h(x)) = g \circ h(x)$.

   (d) Let $F(x) = f(x) + h(\pi x)$. Is F periodic?

   No. In fact, the same exapmle as in part (c) works, by much the same reasoning. However, if you let $F(x) = f(x) + h(x/\pi)$, then you do get a periodic function. In fact,

$$F(x+2\pi) = f(x+2\pi) + h((x+2\pi)/\pi) = f(x) + h(x/\pi + 2) = f(x) + h(x/\pi) = F(x).$$

9. Let f be a non-constant, smooth function on $[0,1]$ such that

$$f'' = \lambda f, \qquad f(0) = 0 = f(1).$$

Show that $\lambda < 0$. (Hint: integrate $\int_0^1 f(x) \cdot f''(x)dx$ by parts.)

Suppose $f'' = \lambda f$. Then

$$\begin{aligned} \lambda \int_0^1 f^2(x)dx &= \int_0^1 (\lambda f(x))f(x)dx = \int_0^1 f(x)f''(x)dx \\ &= f \cdot f'|_0^1 - \int_0^1 (f')^2(x)dx = -\int_0^1 (f')^2(x)dx \end{aligned}$$

In the second to last equality, we integrated by part, and in the last equality, we used the fact that $f(0) = 0 = f(1)$. Also, f is nonconstant, so $\int_0^1 f^2(x)dx > 0$ and $\int_0^1 (f'(x))^2 dx > 0$. Thus we see that

$$\lambda = -\frac{\int_0^1 (f')^2}{\int_0^1 f^2} < 0.$$

10. For which of the PDEs listed below does the principle of superposition hold (i.e. when can you add solutions to obtain another solution)?

In general, the principle of superposition only holds for linear equations.

(a) $u_{tt} = u_{xt}$ yes (linear)

(b) $u_{tt} = u_x - u_t$ yes (linear)

(c) $u_t^2 = u_{xx}$ no (nonlinear)

(d) $u_{tx} = u_{xx} - u_{tt}$ yes (linear)

11. For each of the initial conditions listed below, solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $0 \le x \le \pi$, $t \ge 0$ and the boundary conditions

$$u(0,t) = 0 = u(\pi, t)$$

for all t.

Recall that the solution to the wave equation is given by

$$u(x,t) = \sum_1^\infty \sin(kx)[a_k \cos(kt) + b_k \sin(kt)]$$

where

$$f(x) = \sum_1^\infty a_k \sin(kx) \qquad g(x) = \sum_1^\infty k b_k \sin(kx).$$

Thus we only have to find the Fourier series for f and g and match the coefficients.

(a) $u(x,0) = f(x) = \sin(2x), \partial_t u(x,0) = g(x) = 0$

Here f is already in Fourier series and $g = 0$. So $a_2 = 1$ and the rest of the coefficients are zero. Therefore,

$$u(x,t) = \sin(2x)\cos(2t).$$

(b) $u(x,0) = f(x) = \sin x, \partial_t u(x,0) = g(x) = \cos(3x)$

Again, f and g are already written in Fourier series. So $a_1 = 1$, $b_3 = 1/3$ and the rest of the coefficients are zero. The solution is

$$u(x,t) = \sin x \cos t + \frac{1}{3}\sin(3x)\sin(3t).$$

(c) $u(x,0) = f(x) = x(x-\pi), \partial_t u(x,0) = g(x) = 0$
Fortunately, we have already computes the Fourier sine series for x^2 and x. Recall that

$$x = \sum_1^\infty \frac{2(-1)^{k+1}}{k}\sin(kx) \qquad x^2 = \sum_1^\infty [\frac{2\pi(-1)^{k+1}}{k} + \frac{2((-1)^k-1)}{k^3\pi}]\sin(kx).$$

Then

$$\begin{aligned} f &= x(x-\pi) = x^2 - \pi x = \sum_1^\infty [\frac{2\pi(-1)^{k+1}}{k} + \frac{2((-1)^k-1)}{k^3\pi}]\sin(kx) - \pi\sum_1^\infty \frac{2(-1)^{k+1}}{k}\sin(kx) \\ &= \sum_1^\infty \frac{2((-1)^k-1))}{k^3\pi}\sin(kx) = -\frac{4}{\pi}\sum_{k \text{ odd}} \frac{1}{k^3}\sin(kx). \end{aligned}$$

Then the solution to the wave equation is

$$u(x,t) = -\frac{4}{\pi}\sum_{k \text{ odd}} \frac{1}{k^3}\sin(kx)\cos(kt).$$

(d) $u(x,0) = f(x) = 0, \partial_t u(x,0) = \frac{\tan x}{1+\tan^2 x}$ (Hint: you don't need to integrate anything to compute the Fourier series for g!)
We will use some trig identities:

$$\frac{\tan x}{1+\tan^2 x} = \frac{\tan x}{\sec^2 x} = \frac{\sin x \cdot \cos^2 x}{\cos x} = \sin x\cos x = \frac{1}{2}\sin(2x).$$

Thus $b_2 = 1$ and the rest of the coefficients are zero. The solution is

$$u(x,t) = \sin(2x)\sin(2t).$$

12. The equation of motion for a damped vibrating string is

$$\partial_t^2 u + k\partial_t u = c^2\partial_x^2 u.$$

As in the case of a free vibrating string, we will consider the case where the endpoints of the string are fixed at the same height, the length of the string is L, and look for a solution of the form

$$u(x,t) = v(x)w(t).$$

(a) What are the ODEs which v and w satisfy?
Plug $u(x,t) = v(x)w(t)$ into the equation:

$$v(w'' + kw') = c^2 v'' w.$$

As in the wave equation, we divide both sides by both v and $c^2 w$ to get

$$\frac{v''}{v}(x) = \frac{w'' + kw'}{c^2 w}(t).$$

Because the left hand side is a function of x alone and the right hand side is a function of t alone, they must both be equal to some constant we'll call $-\lambda^2$. Thus we have two ODEs:

$$v'' + \lambda^2 v = 0 \qquad w'' + kw' + \lambda^2 c^2 w = 0.$$

(b) What boundary conditions and/or initial conditions must v and w satisfy?
First look at the conditions for v: we know $u(0,t) = 0 = v(0)w(t)$. So we must have $v(0) = 0$. Similarly, $u(L,t) = 0$ forces $v(L) = 0$.
For w: we have

$$f(x) = u(x,0) = v(x)w(0).$$

Similarly,

$$g(x) = \partial_t u(x,0) = v(x)w'(0).$$

These are initial conditions for the ODE for w.

(c) Can you use these boundary/initial conditions to say anything about the solutions to the ODEs?
This is much like the wave equation, in that the boundary conditions for v rule out most values of λ. Indeed, we see from the condition $v(0) = 0 = v(L)$ that we must have $\lambda = k\pi/L$ for some integer k, and $v(x) = \sin(k\pi x/L)$.
Now that we know what λ is, it is straight-forward to find the general solution for w, and hence solve the equation for a damped, vibrating string.