# Selected Solutions for Homework 6

Math 2280

Nov. 23, 2003

1. (Problem 7.1.7) We want to find the Fourier cosine and sine representation for the function

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall the the representation is given by

$$f(x) = \int_0^\infty [A(\omega)\cos(x\omega) + B(\omega)\sin(x\omega)]d\omega,$$

where

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\cos(y\omega)dy, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\sin(y\omega)dy.$$

First we compute $A$:

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\cos(y\omega)dy = \frac{1}{\pi}\int_0^1 \cos(y\omega)dy = \frac{1}{\pi\omega}\sin(y\omega)|_0^1 \\ &= \frac{\sin\omega}{\pi\omega}. \end{aligned}$$

Next we compute $B$:

$$\begin{aligned} B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\sin(y\omega)dy = \frac{1}{\pi}\int_0^1 \sin(y\omega)dy = -\frac{1}{\pi\omega}\cos(y\omega)|_0^1 \\ &= \frac{1-\cos(\omega)}{\pi\omega}. \end{aligned}$$

Putting this all together, we have

$$f(x) = \int_0^\infty \omega^{-1}[\sin(\omega)\cos(x\omega) + (1-\cos(\omega))\sin(x\omega)]d\omega.$$

2. (Problem 7.1.13)

(a) We know from Example 1 of the text that if

$$f(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

then

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)\cos(\omega x)}{\omega}d\omega.$$

Evaluating this at $x = 1$, which a point of discontinuity of $f$, we see

$$\frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega}{\omega}d\omega = \frac{1}{2}(f(1_+) + f(1_-)) = \frac{1}{2}(1+0) = \frac{1}{2},$$

or

$$\frac{\pi}{4} = \int_0^\infty \frac{\sin\omega\cos\omega}{\omega}d\omega.$$

(b) Now we integrate

$$\int_0^\infty \frac{\sin^2\omega}{\omega^2}d\omega$$

by parts. Let $u = \sin^2\omega$ and $dv = d\omega/\omega^2$. Then

$$du = 2\sin\omega\cos\omega d\omega, \qquad v = -\frac{1}{\omega},$$

so

$$\int_0^\infty \frac{\sin^2\omega}{\omega^2}d\omega = -\frac{\sin^2\omega}{\omega}\Big|_0^\infty + \int_0^\infty \frac{2\cos\omega\sin\omega}{\omega}d\omega = 2\int_0^\infty \frac{\cos\omega\sin\omega}{\omega}d\omega = \frac{\pi}{2}.$$

3. (Problem 7.2.1) We want to compute the Fourier transform of

$$f(x) = \begin{cases} 1 & |x| < 2 \\ 0 & |x| \geq 2. \end{cases}$$

We have

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-ix\omega} dx = \frac{1}{-i\omega\sqrt{2\pi}} e^{-ix\omega}\Big|_{-2}^{2} \\ &= \frac{1}{i\omega\sqrt{2\pi}}(e^{2i\omega} - e^{-2i\omega}) = \frac{1}{i\omega\sqrt{2\pi}}(\cos(2\omega) + i\sin(2\omega) - \cos(2\omega) + i\sin(2\omega)) \\ &= \frac{2\sin(2\omega)}{\omega\sqrt{2\pi}} = \sqrt{2/\pi}\frac{\sin(2\omega)}{\omega}. \end{aligned}$$

4. (Problem 7.2.4) We want to compute the Fourier transform of the function

$$f(x) = \begin{cases} x & |x| > 1 \\ 0 & |x| \geq 1. \end{cases}$$

We have

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} xe^{-ix\omega} dx = \frac{1}{\sqrt{2\pi}}[-\frac{x}{i\omega}e^{-ix\omega}\Big|_{x=-1}^{x=1} + \frac{1}{i\omega}\int_{-1}^{1} e^{-ix\omega}dx] \\ &= \frac{1}{\sqrt{2\pi}}[-\frac{e^{-i\omega}}{i\omega} - \frac{e^{i\omega}}{i\omega} + \frac{1}{\omega^2} e^{-ix\omega}\Big|_{x=-1}^{x=1}] = \frac{1}{\sqrt{2\pi}}[-\frac{2\cos\omega}{i\omega} + \frac{e^{-i\omega} - e^{i\omega}}{\omega^2}] = \frac{1}{\sqrt{2\pi}}[\frac{2i\cos(\omega)}{\omega} + \frac{2\sin(\omega)}{\omega^2}] \\ &= \frac{i\omega^2\sqrt{2}}{\sqrt{\pi}}[\omega\cos(\omega) + \sin(\omega)] \end{aligned}$$

5. (Problem 7.2.10)

We start by recalling the formulas for the Fourier transform and inverse Fourier transform:

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(y)dy, \quad \check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(y)dy.$$

(a) $\hat{f}(x) = \check{f}(-x)$

This is clear from the formulas above. Indeed,

$$\check{f}(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(-x)y} f(y)dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(y)dy = \hat{f}(x).$$

(b) $\mathcal{F}^2(f)(x) = f(-x)$.

Applying the Fourier transform to both sides of the equation of part (a), we have

$$\mathcal{F}^2(f)(x) = \mathcal{F}(\mathcal{F}^{-1}(f))(-x) = f(-x).$$

(c) By definition, $f$ is even if and only if $f(-x) = f(x)$. However, $\mathcal{F}^2(f)(x) = f(-x)$, so $f$ is even if and only if $\mathcal{F}^2(f)(x) = f(x)$. The argument for odd functions is similar.

(d) $\mathcal{F}^4(f) = f$.

This time we just apply part (b) twice. Indeed,

$$\mathcal{F}^4(f)(x) = \mathcal{F}^2(\mathcal{F}^2(f))(x) = f(-(-x)) = f(x).$$