Selected Solutions for Homework 5

Math 2280

Nov. 13, 2003

1. The Laplacian in polar coordinates:

   The book already has a derivation of this formula (c.f. section 4.1), but their derivation looked more complicated than I remember it. So I came up with the following derivation, which I think is a little bit easier to follow. Of course, both derivations are correct.

   We will start with

   $$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

   Below we will let u be any function of two variables, with at least two derivatives. Then

   $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}.$$

   Taking one more derivative, we have

   $$\begin{aligned}\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}[\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}] \\ &= \cos\theta[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial r}] + \sin\theta[\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r}] \\ &= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

   Similarly,

   $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}.$$

   Taking one more derivative, we see

   $$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= r\frac{\partial}{\partial \theta}[-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}] \\ &= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r[-\sin\theta(\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial \theta}) + \cos\theta(\frac{\partial^2 u}{\partial x\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta})] \\ &= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r[-\sin\theta(-r\sin\theta\frac{\partial^2 u}{\partial x^2} + r\cos\theta\frac{\partial^2 u}{\partial x\partial y}) + \cos\theta(-r\sin\theta\frac{\partial^2 u}{\partial x\partial y} + r\cos\theta\frac{\partial^2 u}{\partial y^2})] \\ &= -r\cos\theta\frac{\partial u}{\partial x} - r\sin\theta\frac{\partial u}{\partial y} + r^2[\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}] \\ &= -r\frac{\partial u}{\partial r} + r^2[\sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2}]\end{aligned}$$

   Now we're ready to put everything together:

   $$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \sin^2\theta\frac{\partial^2 u}{\partial x^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} \\ &= (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial x^2} + (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r} \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r},\end{aligned}$$

   which we can rearrange to read

   $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u.$$

2. (Problem 3.8.1)

   We want to solve the boundary value problem

   $$\Delta u = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0, \quad u(x,2) = x.$$

The domain is the rectangle $\{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2\}$. First we separate variables and write $u(x, y) = X(x)Y(y)$. Then the differential equation becomes

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

for some constant λ. We have to have this sign on the separation constant because of the boundary values.

First we treat the equation for X. We have

$$X'' = -\lambda^2 X,$$

so

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

The boundary conditions on X are

$$X(0) = 0 = X(1).$$

Plugging in the left boundary point, we see

$$0 = X(0) = c_1,$$

so we may as well take

$$X(x) = \sin(\lambda x).$$

Here we have absorbed the constnat c_2 into the Y term. Now we plug in the right endpoint, and see that

$$0 = X(1) = \sin(\lambda),$$

so

$$\lambda = \lambda_n = n\pi$$

for $n = 1, 2, 3, \ldots$

Next we treat the equation for Y. This time we have

$$Y'' = \lambda_n^2 Y = n^2\pi^2 Y,$$

which has the solution

$$Y(y) = a_n \cosh(n\pi y) + b_n \sinh(n\pi y).$$

The bottom boundary condition is

$$0 = Y(0) = a_n,$$

so we have $Y(y) = b_n \sinh(n\pi y)$.

Finally, we put everything together and use the boundary condition on the top boundary condition:

$$u(x, y) = \sum b_n \sin(n\pi x) \sinh(n\pi y),$$

where

$$x = u(x, 2) = \sum b_n \sin(n\pi x) \sinh(2n\pi).$$

To find b_n, we need to find the Fourier sine series for the function $x = \sum B_n \sin(n\pi x)$:

$$\begin{aligned} B_n &= 2\int_0^1 x \sin(n\pi x) dx = 2[-\frac{x}{n\pi} \cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x) dx] \\ &= -\frac{2\cos(n\pi)}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}. \end{aligned}$$

Matching the Fourier coefficients, we see that

$$b_n = \frac{B_n}{\sinh(n\pi)} = \frac{2(-1)^{n+1}}{n\pi \sinh(n\pi)},$$

and so

$$u(x, y) = \sum \frac{2(-1)^{n+1}}{n\pi \sinh(n\pi)} \sin(n\pi x) \sinh(n\pi y).$$

3. (Problem 4.1.1)

   We want to compute the Lapcian of

$$u(x,y) = \frac{x}{x^2+y^2} = \frac{r\cos(\theta)}{r^2} = r^{-1}\cos(\theta).$$

   We have

$$\Delta u = \partial_r^2(r^{-1}\cos\theta) + r^{-1}\partial_r(r^{-1}\cos\theta) + r^{-2}\partial_\theta^2(r^{-1}\cos\theta) = 2r^{-3}\cos\theta - r^{-3}\cos\theta - r^{-3}\cos\theta = 0.$$

4. (Problem 4.1.3)

   We want to compute the Laplacian of

$$u(x,y) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}.$$

   We have

$$\begin{aligned}\Delta u &= \partial_r^2 u + r^{-1}\partial_r u + r^{-2}\partial_\theta^2 u = \partial_r^2(r^{-1}) + r^{-1}\partial_r(r^{-1}) \\ &= 2r^{-3} - r^{-3} = r^{-3} = \frac{1}{(x^2+y^2)^{-3/2}}.\end{aligned}$$

5. (Problem 4.1.6)

   We want to compute the Laplacian of

$$u(x,y) = \ln(x^2+y^2) = \ln(r^2).$$

   Then we have

$$\begin{aligned}\Delta u &= \partial_r^2 u + r^{-1}\partial_r u + r^{-2}\partial_\theta^2 u = \partial_r^2(\ln(r^2)) + r^{-1}\partial_r(\ln(r^2)) \\ &= -2r^{-2} + 2r^{-2} = 0.\end{aligned}$$

6. (Problem 4.4.2)

   We want to find the harmonic function (i.e. $\Delta u = 0$) such that

$$u(1,\theta) = f(\theta) = \sin(2\theta).$$

   The genereal form of our solution is

$$u(r,\theta) = a_0 + \sum r^n[a_n\cos(n\theta) + b_n\sin(n\theta)],$$

   where

$$f(\theta) = a_0 + \sum[a_n\cos(n\theta) + b_n\sin(n\theta)].$$

   Fortunately, our boundary data f is already in Fourier series. Matching the Fourier coefficients, we see that

$$b_2 = 1$$

   and all the other coefficients are zero. Therefore,

$$u(r,\theta) = r^2\sin(2\theta).$$

7. (Problem 4.4.5)

   We want to find a harmonic function (i.e. $\Delta u = 0$) on the unit disc with the boundary data

$$u(1,\theta) = f(\theta) = \begin{cases} 100 & 0 \le \theta \le \pi/4 \\ 0 & \pi/4 < \theta < 2\pi. \end{cases}$$

   The general form of our solution is

$$u(r,\theta) = a_0 + \sum r^n[a_n\cos(n\theta) + b_n\sin(n\theta)],$$

where

$$f(\theta) = a_0 + \sum [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

As usual, the work in this problem is to find the Fourier series of the boundary data. We start with a_0:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = \frac{100}{2\pi} \int_0^{\pi/4} d\theta = \frac{25}{2}.$$

Next we compute

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta = \frac{100}{\pi} \int_0^{\pi/4} \cos(n\theta) d\theta = \frac{100}{n\pi} \sin(n\theta) \Big|_0^{\pi/4} \\ &= \frac{100 \sin(n\pi/4)}{n\pi}. \end{aligned}$$

One can write this expression in closed form, depending on the remainder of $n/4$, but it's ok as is. Finally, we compute

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta = \frac{100}{\pi} \int_0^{\pi/4} \sin(n\theta) d\theta = -\frac{100}{n\pi} \cos(n\theta) \Big|_0^{\pi/4} \\ &= \frac{100}{n\pi} (1 - \cos(n\pi/4)). \end{aligned}$$

Finally, we can put everything together to get

$$u(r, \theta) = \frac{25}{2} + \sum \frac{100 r^n}{n\pi} [\sin(n\pi/4) \cos(n\theta) + (1 - \cos(n\pi/4)) \sin(n\theta)].$$