Selected Solutions for Homework 4

Math 2280

Oct. 31, 2003

1. (Problem 3.5.2)

   We want to solve the heat equation

$$\partial_t u = \partial_x^2 u, \quad u(0,t) = 0 = u(\pi,t), u(x,0) = 30\sin(x).$$

   We know that the general form of the solution is

$$u(x,t) = \sum a_n e^{-n^2 t}\sin(nx),$$

   where a_n are the Fourier coefficients of $u(x,0)$. However, the initial data is already in Fourier series, so we can just red off these coefficients: $a_1 = 30$ and all the other coefficients are zero. Thus

$$u(x,t) = 30e^{-t}\sin(x).$$

2. (Problem 3.5.10) We want to compute the Fourier coefficients for the solution to the heat equation in the case of inhomogeneous boundary data. Here we have

$$u(0,t) = T_1, \qquad u(L,t) = T_2, \qquad u(x,0) = f(x).$$

   The steady state temperature u_0 is the linear function interpolating the two boundary data points, which is

$$u_0(x) = T_1 + \frac{(T_2 - T_1)x}{L}.$$

   Then the initial data for the modified problem is

$$f(x) - u_0(x) = f(x) - T_1 + \frac{(T_2 - T_1)x}{L}.$$

   This means that the Fourier coefficients are given by

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L [f(x) - T_1 + \frac{(T_2 - T_1)x}{L}]\sin((n\pi x)/L)dx \\
&= \frac{2}{L}\int_0^L f(x)\sin((n\pi x)/L)dx - \frac{2}{L}\int_0^L [T_1 + \frac{(T_2 - T_1)x}{L}]\sin((n\pi x)/L)dx \\
&= \frac{2}{L}\int_0^L f(x)\sin((n\pi x)/L)dx - \frac{2T_1}{L}\int_0^L \sin((n\pi x)/L)dx + \frac{2(T_2 - T_1)}{L^2}\int_0^L x\sin((n\pi x)/L)dx \\
&= \frac{2}{L}\int_0^L f(x)\sin((n\pi x)/L)dx \\
&\quad + \frac{2T_1}{L}\cdot\frac{L}{n\pi}\cos((n\pi x)/L)|_0^L + \frac{2(T_2 - T_1)}{L^2}[-\frac{Lx}{n\pi}\cos((n\pi x)/L)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos((n\pi x)/L)dx] \\
&= \frac{2}{L}\int_0^L f(x)\sin((n\pi x)/L)dx + \frac{2T_1}{n\pi}((-1)^n - 1) + \frac{2(T_2 - T_1)(-1)^n}{n\pi} \\
&= \frac{2}{L}\int_0^L f(x)\sin((n\pi x)/L)dx - 2\frac{T_1 + (-1)^{n+1}T_2}{n\pi}.
\end{aligned}$$

3. (Problem 3.6.2)

   We want to solve the heat equation

$$\partial_t u = \partial_x^2 u, \quad \partial_x u(0,t) = 0 = \partial_x u(\pi,t), \quad u(x,0) = x.$$

   The general solution is given by

$$u(x,t) = a_0 + \sum a_n e^{-n^2 t}\cos(nx)$$

   where

$$u(x,0) = a_0 + \sum a_n \cos(nx).$$

We have to find the Fourier coefficients for the function $f(x) = x$. First,

$$a_0 = \frac{1}{\pi}\int_0^\pi x dx = \frac{x^2}{2\pi}\bigg|_0^\pi = \frac{\pi}{2}.$$

Then for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int +0^\pi x\cos(nx)dx = \frac{2}{\pi}[\frac{x}{n}\sin(nx)\bigg|_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)dx] \\ &= -\frac{2}{n\pi}\int_0^\pi \sin(nx)dx = \frac{2}{n^2\pi}\cos(nx)|_0^\pi = \frac{2}{n^2\pi}((-1)^n - 1). \end{aligned}$$

This last quantity is 0 for n even and $-4/(n^2\pi)$ for n odd. So we have

$$u(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos(nx)e^{-n^2 t}.$$

4. (Problem 3.6.7)

We have a solution u to a heat equation with insulated ends, so

$$\partial_t u = c^2 \partial_x^2 u, \qquad \partial_x u(0,t) = 0 = \partial_x u(L,t).$$

Let

$$U(t) = \frac{1}{L}\int_0^L u(x,t)dx$$

denote the average temperature at time t. There are two ways to show that U is constant: one can use the hint in the book or one can use the differential equation.

First I'll present the solution which doesn't follow the hint in the book. We have

$$\begin{aligned} \frac{dU}{dt} &= \frac{1}{L}\cdot\frac{d}{dt}\int_0^L u(x,t)dx = \frac{1}{L}\int_0^L \partial_t u(x,t)dx \\ &= \frac{c^2}{L}\int_0^L \partial_x^2 u(x,t)dx = \frac{c^2}{L}\partial_x u(x,t)|_{x=o}^{x=L} = 0. \end{aligned}$$

Here we have used the fact that u is bounded and L is finite to differentiate underneath the integral sign. We have also used the fundamental theorem of calculus and the boundary conditions on u.

There is another solution, where you use the fact that the general solution for u is given by

$$u(x,t) = a_0 + \sum_1^\infty a_n e^{-n^2 t}\cos((n\pi x)/L).$$

Then evaluating U, we find

$$\begin{aligned} U(t) &= \frac{1}{L}\int_0^L u(x,t)dx = \frac{1}{L}\int_0^L [a_0 + \sum a_n e^{-n^2 t}\cos((n\pi x)/L)dx \\ &= \frac{1}{L}\int_0^L a_0 dx + \sum \frac{e^{-n^2 t}}{L}\int_0^L \cos((n\pi x)/L)dx = a_0 + \frac{e^{-n^2 t}}{L}(-\frac{L}{n\pi})\sin((n\pi x)/L)\bigg|_0^L = a_0. \end{aligned}$$

One can justify bringing the sum out of the integral using the fact that the series a_n is in l^2, i.e. $\sum a_n^2 < \infty$, to get an a priori bound. More precisely,

$$|u(x,t)| \leq \sum |a_n| e^{-n^2 t},$$

which is bounded independent of t.

Of course, both solutions are correct. However, I find the first solution more satisfying because one does not need to know anything about the general form of the solution. This is really just my personal taste.

5. (Problem 3.7.1)

We want to solve the wave equation

$$\partial_t^2 u = \frac{1}{\pi}(\partial_x^2 u + \partial_y^2 u), \quad u(x,y,0) = \sin(3\pi x)\sin(\pi y), \partial_t u(x,y,0) = 0, \quad u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0.$$

The general form of the solution is given by

$$u(x,y,t) = \sum_{m,n} \sin(m\pi x)\sin(n\pi y)[a_{m,n}\cos(\sqrt{n^2+m^2}t) + b_{m,n}\sin(\sqrt{n^2+m^2}t)].$$

Evaluating at $t=0$, we see that the coefficients are given by

$$u(x,y,0) = \sum_{m,n} a_{m,n}\sin(m\pi x)\sin(n\pi y), \qquad \partial_t u(x,y,0) = \sum_{m,n} b_{m,n}\sqrt{m^2+n^2}\sin(m\pi x)\sin(n\pi y).$$

Because $\partial_t u(x,y,0) = 0$, we see that the coefficients $b_{m,n}$ are all zero. Also, the initial data $u(x,y,0)$ is already in Fourier series, so $a_{3,1} = 1$ and the rest of the coefficients are zero. Thus we have

$$u(x,y,t) = \sin(3\pi x)\sin(\pi y)\cos(\sqrt{10}t).$$

6. (Problem 3.7.9)

We want to find the nodla lines for the function

$$u(x,y,t) = \sin(4\pi x)\sin(\pi y)\cos(t).$$

These are the lines which stay at the equilibrium position for all time; in other words, we are looking for the (x,y) such that

$$u(x,y,t) = 0 \text{ for all } t.$$

Thus we must

$$0 = \sin(4\pi x)\sin(\pi x), \qquad 0 \le x \le 1, \qquad 0 \le y \le 1.$$

This forces either $\sin(\pi y) = 0$, in which case $y = 0, 1$, or $\sin(4\pi x) = 0$, in which case $x = 0, 1/4/, 1/2, 3/4, 1$. So the nodal lines are the boundary of the square, together with the lines $x = 1/4$, $x = 1/2$, and $x = 3/4$. Here is a picture of them: