

Selected Solutions for Homework 3

Math 2280
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1. (Problem 3.3.3a)

We want to solve

$$\partial_t^2 u = \partial_x^2 u, \quad u(0, t) = 0 = u(1, t), \quad u(x, 0) = \sin(\pi x) + 3 \sin(2\pi x) - \sin(5\pi x), \quad \partial_t u(x, 0) = 0.$$

We know that the general form of the solution is

$$u(x, t) = \sum_n [\sin(n\pi x)(\alpha_n \cos(n\pi t) + \beta_n \sin(n\pi t))],$$

where

$$u(x, 0) = \sum_n \alpha_n \sin(n\pi x), \quad \partial_t u(x, 0) = \sum_n n\beta_n \sin(n\pi x).$$

Matching the coefficients with the given initial position and initial velocity, we see that

$$\alpha_1 = 1, \quad \alpha_2 = 3, \quad \alpha_5 = -1,$$

and all the other coefficients are zero. Therefore, the solution is

$$u(x, t) = \sin(\pi x) \cos(\pi t) + 3 \sin(2\pi x) \cos(2\pi t) - \sin(5\pi x) \cos(5\pi t).$$

2. (Problem 3.3.12)

Here we're starting with the equation

$$\partial_t^2 u + 2k\partial_t u = c^2 \partial_x^2 u, \quad u(0, t) = 0 = u(L, t), \quad u(x, 0) = f(x), \quad \partial_t u(x, 0) = g(x).$$

(a) First we separate variables: look for a solution in the form

$$u(x, t) = X(x)T(t).$$

Then the equation becomes

$$XT'' + 2kXT' = c^2 X''T,$$

which we can rearrange to read

$$\frac{T'' + 2kT'}{c^2 T}(t) = \frac{X''}{X}(x).$$

Notice that the right hand side is a function of x alone while the left hand side is a function of t alone. Therefore, both sides of the equation must be equal to some constant, which we will call $-\mu^2$. The differential equation then becomes the following two ODEs:

$$T'' + 2kT' + \mu^2 c^2 T = 0, \quad X'' + \mu^2 X = 0.$$

(b) Let's examine the equation for X first. Recall that we have boundary conditions $u(0, t) = 0 = u(L, t)$, which implies

$$X(0) = 0 = X(L).$$

The general solution of the equation $X'' + \mu^2 X = 0$ is

$$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Matching the boundary condition at $x = 0$, we get $0 = X(0) = c_1$, so we can just take

$$X(x) = \sin(\mu x).$$

Now we need to match the boundary condition at $x = L$. We have

$$0 = X(L) = \sin(\mu L),$$

so $\mu L = n\pi$ for $n = 1, 2, 3, \dots$. We can rearrange this to read

$$\mu = \frac{n\pi}{L}.$$

(c) Now we turn our attention to the equation for T :

$$T'' + 2kT' + \mu^2 c^2 T = 0.$$

This is a second order, linear ODE with constant coefficients. If we try solutions of the form $T(t) = e^{rt}$ then we find

$$r^2 + 2kr + \mu^2 c^2 = 0.$$

Then by the quadratic formula

$$r = \frac{-2k \pm \sqrt{4k^2 - 4\mu^2 c^2}}{2} = -k \pm \sqrt{k^2 - \mu^2 c^2} = -k \pm \sqrt{k^2 - \frac{c^2 n^2 \pi^2}{L^2}}.$$

Here we have used the formula we found for μ . The form of the solution depends on the sign of the discriminant $k^2 - (c^2 n^2 \pi^2)/L^2$. Regardless of the sign, we will let

$$\lambda_n = \sqrt{|k^2 - (c^2 n^2 \pi^2)/L^2|}.$$

If $k^2 - (c^2 n^2 \pi^2)/L^2 > 0$ (i.e. $k > (cn\pi)/L$), then the solutions are exponential, and

$$T(t) = e^{-kt} [c_1 e^{\lambda_n t} + c_2 e^{-\lambda_n t}].$$

This is equivalent to the solution written in the book; recall that $\cosh t = (1/2)(e^t + e^{-t})$ and $\sinh t = (1/2)(e^t - e^{-t})$. If $k^2 - (c^2 n^2 \pi^2)/L^2 < 0$ (i.e. $k < (cn\pi)/L$) then the solutions are oscillatory, and

$$T(t) = e^{-kt} [c_1 \cos(\lambda_n t) + c_2 \sin(\lambda_n t)].$$

If $k^2 - (c^2 n^2 \pi^2)/L^2 = 0$ (i.e. $k = (cn\pi)/L$) then $\lambda_n = 0$ and the solutions are of the form

$$T(t) = e^{-kt} [c_1 + c_2 t].$$

Notice that this last case can only occur if $k = (cn\pi)/L$ for some positive integer n , or equivalently if $(kL)/(c\pi)$ is a positive integer.

(d) In the case where $(kL)/(c\pi)$ is not a positive integer, we only have the exponential and oscillatory solutions. When $n < (kL)/(c\pi)$, we have $k > (cn\pi)/L$, and so the solutions are exponential. Then $n > (kL)/(c\pi)$, we have $k < (cn\pi)/L$, and so the solutions are oscillatory. By the superposition principle, the general solution is a sum of all these solutions, summing over n . So we have

$$\begin{aligned} u(x, t) &= e^{-kt} \sum_{n < (kL)/(c\pi)} \sin((n\pi x)/L) [a_n \cosh(\lambda_n t) + b_n \sinh(\lambda_n t)] \\ &\quad + e^{-kt} \sum_{n > (kL)/(c\pi)} \sin((n\pi x)/L) [a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)]. \end{aligned}$$

Now we need to match the initial conditions. Evaluating the sum at $t = 0$, we obtain

$$\begin{aligned} u(x, 0) &= \sum_{n < (kL)/(c\pi)} \sin((n\pi x)/L) [a_n \cosh(0) + b_n \sinh(0)] + \sum_{n > (kL)/(c\pi)} \sin((n\pi x)/L) [a_n \cos(0) + b_n \sin(0)] \\ &= \sum_n a_n \sin((n\pi x)/L) = f(x). \end{aligned}$$

Thus the coefficients a_n give the Fourier sine series for f and

$$a_n = \frac{2}{L} \int_0^L f(x) \sin((n\pi x)/L) dx.$$

Next we need to evaluate the derivative of the solution at $t = 0$:

$$\begin{aligned} \partial_t u(x, 0) &= -k \sum_{n < (kL)/(c\pi)} \sin((n\pi x)/L) [a_n \cosh(0) + b_n \sinh(0)] \\ &\quad + \sum_{n < (kL)/(c\pi)} \lambda_n \sin((n\pi x)/L) [a_n \sinh(0) + b_n \cosh(0)] \\ &\quad -k \sum_{n > (kL)/(c\pi)} \sin((n\pi x)/L) [a_n \cos(0) + b_n \sin(0)] \\ &\quad + \sum_{n > (kL)/(c\pi)} \lambda_n \sin((n\pi x)/L) [-a_n \sin(0) + b_n \cos(0)] \\ &= -k \sum a_n \sin((n\pi x)/L) + \sum \lambda_n b_n \sin((n\pi x)/L) = \sum (-ka_n + \lambda_n b_n) \sin((n\pi x)/L) = g(x). \end{aligned}$$

Thus the coefficients $-ka_n + \lambda_n b_n$ give the Fourier sine series of g and

$$-ka_n + \lambda_n b_n = \frac{2}{L} \int_0^L g(x) \sin((n\pi x)/L) dx.$$

(e) In the last case where $(kL)/c\pi$ is a positive integer, we just have to add the $n = (kL)/(c\pi)$ case to our general solution above. This Fourier mode of the solution is

$$\sin((kx)/c)[a_n e^{-kt} + b_n t e^{-kt}].$$

Evaluating at $t = 0$ we have

$$u(x, 0) = a_n \sin((kx)/c),$$

so

$$a_n = \frac{2}{L} \int_0^L f(x) \sin((n\pi x)/L) dx$$

as before. Evaluating the derivative at $t = 0$ we have

$$\partial_t u(x, 0) = \sin((kx)/c)[-ka_n + b_n],$$

so this time we have

$$-ka_n + b_n = \frac{2}{L} \int_0^L g(x) \sin((n\pi x)/L) dx.$$

3. (Problem 3.4.1)

We want to solve

$$\partial_t^2 u = \frac{1}{\pi^2} \partial_x^2 u, \quad u(0, t) = 0 = u(1, t), \quad u(x, 0) = \sin(\pi x), \quad \partial_t u(x, 0) = 0$$

using D'Alembert's method. In this case, $\sin \pi x$ is already 2-periodic and odd, so we have the right extension of the initial position $u(x, 0) = f(x)$. The initial velocity is zero, so we don't have to worry about it. Our solution in this case is

$$u(x, t) = \frac{1}{2}[f(x + t/\pi) + f(x - t/\pi)] = \frac{1}{2}[\sin(\pi x + t) - \sin(\pi x - t)].$$

4. (Problem 3.4.13)

We are given that the initial position f satisfies $f(x) = f(L-x)$ and the initial velocity g satisfies $g(x) = g(L-x)$. Recall that D'Alembert's solution to the wave equation is given by

$$u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Let's first examine the part of the solution coming from the initial position:

$$\begin{aligned} f(x + c(t + L/c)) + f(x - c(t + L/c)) &= f(x + ct + L) + f(x - ct - L) \\ &= f(L - (x + ct + L)) + f(L - (x - ct - L)) \\ &= f(-x - ct) + f(2L - x + ct) \\ &= -f(x + ct) + f(-x + ct) = -f(x + ct) - f(x - ct). \end{aligned}$$

Here we have used the fact that f is odd and $2L$ -periodic. For the other part of the solution, we have

$$\begin{aligned} \int_{x-ct-L}^{x+ct+L} g(s) ds &= \int_{x-ct-L}^{x+ct+L} g(L-s) ds = - \int_{2L-x+ct}^{-x-ct} g(\tilde{s}) d\tilde{s} \\ &= - \int_{2L-x+ct}^{-x+ct} g(\tilde{s}) d\tilde{s} - \int_{-x+ct}^{-x-ct} g(\tilde{s}) d\tilde{s} = \int_{x+ct}^{-x-ct} g(\tilde{s}) d\tilde{s} \\ &= - \int_{x-ct}^{x+ct} g(\tau) d\tau. \end{aligned}$$

Here we have used the change of variables $\tilde{s} = L - s$ and $\tau = -\tilde{s}$. Because the integral does not depend on the variable of integration, we have just shown

$$\int_{x-ct-L}^{x+ct+L} g(s) ds = - \int_{x-ct}^{x+ct} g(s) ds.$$

Putting this all together, we see that

$$\begin{aligned}u(x, t + L/c) &= \frac{1}{2}[f(x + ct + L) + f(x - ct - L)] + \frac{1}{2c} \int_{x-ct-L}^{x+ct+L} g(s) ds \\ &= -\frac{1}{2}[f(x + ct) + f(x - ct)] - \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds = -u(x, t).\end{aligned}$$