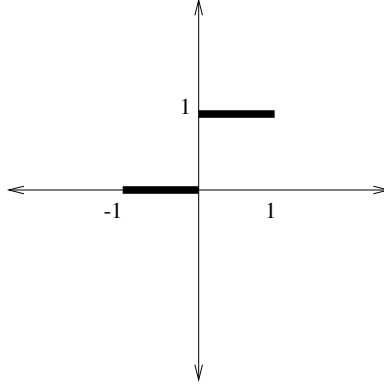


Selected Solutions for Homework 2  
Math 2280  
Sept. 22, 2003

1. (Problem 2.3.13)

We have a function  $f$  given by the graph below.



This function is 1 for  $0 < x < 1$  and 0 for  $-1 < x < 0$ . There are two ways we can compute the Fourier series: either by direct computation or by using the Fourier series in problem 1. First we will use the Fourier series in problem 1. Let  $F$  be defined by

$$F(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}.$$

Then  $F$  has a Fourier series

$$F(x) = \frac{4}{\pi} \sum_0^{\infty} \frac{1}{2k+1} \sin((2k+1)\pi x).$$

Now notice that

$$f(x) = \frac{1}{2}(F(x) + 1) = \frac{1}{2} \left[ \frac{4}{\pi} \sum_0^{\infty} \frac{1}{2k+1} \sin((2k+1)\pi x) + 1 \right] = \frac{1}{2} + \frac{2}{\pi} \sum_0^{\infty} \frac{1}{2k+1} \sin((2k+1)\pi x).$$

For the sake of completeness, we will actually compute the Fourier coefficients. First,

$$a_0 = \frac{1}{2} \int_{-1}^1 f(x) dx = \frac{1}{2} \int_0^1 dx = \frac{1}{2}.$$

Next, for  $k \geq 1$  we have

$$a_k = \int_{-1}^1 f(x) \cos(k\pi x) dx = \int_0^1 \cos(k\pi x) dx = 0.$$

Finally, we have

$$b_k = \int_{-1}^1 f(x) \sin(k\pi x) dx = \int_0^1 \sin(k\pi x) dx = -\frac{1}{k\pi} \cos(k\pi x) \Big|_0^1 = \frac{1}{k\pi} (1 - (-1)^k).$$

This last coefficient is 2 if  $k$  is odd and 0 if  $k$  is even. Thus we have

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k \text{ odd}} \frac{1}{k} \sin(k\pi x) = \frac{1}{2} + \frac{2}{\pi} \sum_0^{\infty} \frac{1}{2k+1} \sin((2k+1)\pi x).$$

2. (Problem 2.3.17)

We start with the Fourier series

$$x^2 = \frac{p^2}{3} + \frac{4p^2}{\pi} \sum_1^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi x/p) / k^2$$

for  $-p \leq x \leq p$ .

(a) Evaluating this series at  $x = 0$ , we find

$$0 = \frac{p^2}{3} + \frac{4p^2}{\pi} \sum_1^{\infty} \frac{(-1)^k}{k^2}.$$

Substituting  $p = \pi/2$  we find

$$0 = \frac{\pi^2}{12} + \pi \sum_1^{\infty} \frac{(-1)^k}{k^2},$$

which we can rearrange to read

$$\frac{\pi}{12} = \sum_1^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

(b) Now we return to the Fourier series we were given and evaluate at  $x = p$  to obtain (using  $\cos(k\pi) = (-1)^k$ )

$$p^2 = \frac{p^2}{3} + \frac{4p^2}{\pi} \sum_1^{\infty} \frac{1}{k^2}.$$

We can rearrange this series to read

$$\frac{\pi}{6} = \sum_1^{\infty} \frac{1}{k^2}.$$

Adding this series to the series we obtained in part (a) we have

$$\frac{\pi}{4} = \frac{3\pi}{12} = \frac{\pi}{12} + \frac{\pi}{6} = (1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots) + (1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots) = 2(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots).$$

Dividing by 2, we have

$$\frac{\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

### 3. (Problem 2.3.25)

We have a  $2p$ -periodic function  $f$ , which is piecewise smooth and continuous. We are also suppose to assume that  $f'$  is piecewise smooth. Let  $f$  and  $f'$  have the Fourier series:

$$f(x) = a_0 + \sum [a_k \cos(k\pi x/p) + b_k \sin(k\pi x/p)] \quad f'(x) = a'_0 + [a'_k \cos(k\pi x/p) + b'_k \sin(k\pi x/p)].$$

We are to derive relationships between the Fourier coefficients for  $f$  and those for  $f'$ .

First let's look at  $a'_0$ :

$$a'_0 = \frac{1}{2p} \int_{-p}^p f'(x) dx = \frac{1}{2p} f(x)|_{-p}^p = 0.$$

Next we look at  $a'_k$ :

$$\begin{aligned} a'_k &= \frac{1}{p} \int_{-p}^p f'(x) \cos(k\pi x/p) dx = f(x) \cos(k\pi x/p)|_{-p}^p + \frac{1}{p} \int_{-p}^p f(x) \cdot \frac{k\pi}{p} \cdot \sin(k\pi x/p) dx \\ &= [f(p) \cos(k\pi) - f(-p) \cos(-k\pi)] + \frac{k\pi}{p} \cdot \frac{1}{p} \int_{-p}^p f(x) \sin(k\pi x/p) dx = \frac{k\pi}{p} b_k. \end{aligned}$$

Here we have integrated by parts, using  $\cos(k\pi x/p) = u$  and  $f' dx = dv$ . We have also used the formula for the Fourier coefficient  $b_k$ .

Similarly, we have

$$\begin{aligned} b'_k &= \frac{1}{p} \int_{-p}^p f'(x) \sin(k\pi x/p) dx = f(x) \sin(k\pi x/p)|_{-p}^p - \frac{k\pi}{p} \cdot \frac{1}{p} \int_{-p}^p f(x) \cos(k\pi x/p) dx \\ &= -\frac{k\pi}{p} a_k. \end{aligned}$$

4. (Problem 2.3.32) (Remark: problem 26 of this section is essentially the same as problem 25; I didn't think it would be good to assign the same problem twice.)

Here we have the sawtooth function, with Fourier series

$$f(x) = \sum \frac{1}{k} \sin(kx).$$

- (a)  $f$  is piecewise smooth (piecewise linear, in fact) and it is  $2\pi$ -periodic. Also,  $f'$  is piecewise constant, so it is certainly piecewise smooth. However,  $f$  is not continuous at  $x = 2k\pi$ , for any integer  $k$  (the left and right limits disagree).
- (b) Differentiating the Fourier series term by term, we get

$$\sum \cos(kx).$$

Evaluating this sum at zero, we get  $1 + 1 + 1 + \dots$ , which diverges to  $\infty$ . In general,

$$\lim_{k \rightarrow \infty} \cos kx \neq 0,$$

which means the sum  $\sum \cos kx$  cannot converge. This is easiest to see if  $x = p\pi/q$ , where  $p$  and  $q$  are integers (i.e. if  $x$  is a rational multiple of  $\pi$ ). Then if  $k = 2nq$  for any integer  $n$ ,

$$\cos kx = \cos 2np\pi = 1,$$

so  $\cos kx$  cannot tend to zero as  $k \rightarrow \infty$ . To complete the argument, use the fact that rational multiples of  $\pi$  are dense in the real numbers.

5. (Problem 2.4.6a)

We want to compute both of the half-range Fourier expansions for

$$f(x) = \cos x \quad 0 < x < \pi.$$

The even half-range expansion is just  $\cos x$ . The easiest way to see this is to note that  $f$  is already a sum of  $\cos$  functions with the right period, so it is already written in Fourier series. But one can actually do the computation. The function  $\cos x$  has mean zero between 0 and  $\pi$ , so  $a_0 = 0$ . To see that  $a_k = 0$  for  $k \geq 2$ , use the identity

$$\cos(x) + \cos(kx) = \frac{1}{2}[\cos((k+1)x) + \cos((k-1)x)].$$

To see that  $a_1 = 1$ , use the identity

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x)).$$

Next we have to compute the odd half-range expansion. This will have the form

$$\cos x = \sum b_k \sin(kx) \quad \text{where} \quad b_k = \frac{2}{\pi} \int_0^\pi \cos(x) \sin(kx) dx.$$

This time we will have to use the identity

$$\cos(x) \sin(kx) = \frac{1}{2}[\sin((k+1)x) + \sin((k-1)x)].$$

Then

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_0^\pi [\sin((k+1)x) + \sin((k-1)x)] dx = \frac{1}{\pi} \left[ -\frac{1}{k+1} \cos((k+1)x) \Big|_0^\pi - \frac{1}{k-1} \cos((k-1)x) \Big|_0^\pi \right] \\ &= \frac{1}{\pi} \left[ -\frac{1}{k+1} ((-1)^{k+1} - 1) - \frac{1}{k-1} ((-1)^{k-1} - 1) \right] = \frac{1 + (-1)^k}{\pi} \left[ \frac{1}{k+1} + \frac{1}{k-1} \right] \\ &= \frac{2k(1 + (-1)^k)}{\pi(k^2 - 1)}. \end{aligned}$$

Thus the Fourier coefficient is 0 if  $k$  is odd and  $4k/(\pi(k^2 - 2))$  if  $k$  is even and the Fourier series is

$$\cos x = \frac{4}{\pi} \sum_{k \text{ even}} \frac{k}{k^2 - 1} \sin kx.$$

6. (Problem 2.4.11)

The sine series expansion of  $f(x) = \sin(\pi x)$  for  $0 < x < 1$  is just  $\sin(\pi x)$ . The reasoning is very similar to the first part of problem 2.4.6.

7. (Problem 2.4.17)

We have a function  $f$  with the graph shown in the textbook.

- (a) First we derive a formula for  $f$ . Notice that  $f$  is piecewise linear, and the endpoints of the line segments are  $(0, 0)$ ,  $(a, h)$  and  $(p, 0)$  (in order). The first line segment goes through the origin and has slope  $h/a$ . Thus we must have

$$f(x) = \frac{hx}{a} \quad 0 \leq x \leq a.$$

The other line segment has slope  $-h/(p-a) = h/(a-p)$ , and so

$$f(x) = \frac{hx}{a-p} + b \quad a \leq x \leq p.$$

Matching the function values,  $f(a) = h$ , we see

$$b = h - \frac{ah}{a-p} = \frac{h(a-p) - ah}{a-p} = -\frac{hp}{a-p}, \quad f(x) = \frac{h(x-p)}{a-p} \quad a \leq x \leq p.$$

- (b) Now we want to find the sine series representation for  $f$ . We have that

$$f(x) = \sum_1^{\infty} b_k \sin(k\pi x/p)$$

where

$$b_k = \frac{2}{p} \int_0^p f(x) \sin(k\pi x/p) dx = \frac{2}{p} \int_0^a \frac{hx}{a} \sin(k\pi x/p) dx + \frac{2}{p} \int_a^p \frac{h(x-p)}{a-p} \sin(k\pi x/p) dx$$

We will do these two integrals separately. First,

$$\begin{aligned} \frac{2}{p} \int_0^a \frac{hx}{a} \sin(k\pi x/p) dx &= \frac{2h}{pa} \left[ -\frac{p}{k\pi} x \cos(k\pi x/p) \Big|_0^a + \frac{p}{k\pi} \int_0^a \cos(k\pi x/p) dx \right] \\ &= \frac{2h}{ak\pi} \left[ -a \cos(k\pi a/p) + \frac{p}{k\pi} \sin(k\pi a/p) \right] = -\frac{2h}{k\pi} \cos(k\pi a/p) + \frac{2hp}{ak^2\pi^2} \sin(k\pi a/p). \end{aligned}$$

Next we have

$$\begin{aligned} \frac{2}{p} \int_a^p \frac{h(x-p)}{a-p} \sin(k\pi x/p) dx &= \frac{2h}{p(a-p)} \left[ -\frac{p}{k\pi} x \cos(k\pi x/p) \Big|_a^p + \frac{p}{k\pi} \int_a^p \cos(k\pi x/p) dx - p \int_a^p \sin(k\pi x/p) dx \right] \\ &= \frac{2h}{p(a-p)} \left[ -\frac{p^2}{k\pi} \cos(k\pi) + \frac{pa}{k\pi} \cos(k\pi a/p) + \frac{p^2}{k\pi} \cos(k\pi) \right. \\ &\quad \left. - \frac{p^2}{k\pi} \cos(k\pi a/p) + \frac{p^2}{k^2\pi^2} \sin(k\pi x/p) \Big|_a^p \right] \\ &= \frac{2h}{p(a-p)} \left[ \frac{p(a-p)}{k\pi} \cos(k\pi a/p) - \frac{p^2}{k^2\pi^2} \sin(k\pi a/p) \right] \\ &= \frac{2h}{k\pi} \cos(k\pi a/p) - \frac{2hp}{k^2\pi^2(a-p)} \sin(k\pi a/p) \end{aligned}$$

Putting this all together, we get

$$b_k = \sin(k\pi a/p) \left[ \frac{2hp}{ak^2\pi^2} - \frac{2hp}{(a-p)k^2\pi^2} \right] = \frac{2hp^2}{a(p-a)k^2\pi^2} \sin(k\pi a/p).$$

Thus the Fourier series of  $f$  is

$$f(x) = \frac{2hp^2}{a(p-a)\pi^2} \sum_1^{\infty} \frac{1}{k^2} \sin(k\pi a/p) \sin(k\pi x/p).$$

8. (Problem 3.2.6)

In this problem, we want to model the motion of a hanging chain. The top end of the chain is anchored at the position  $x = L, u = 0$  and the density  $\rho$  is constant. See the figure in the text for a picture.

- (a) In the equilibrium position, the chain hangs along the vertical line  $u = 0$ . The piece of chain located at position  $(x, 0)$  supports a mass equal to  $\rho x$ . This is because all the pieces of chain below this particular piece pull down on this piece. As the only force acting on the chain is gravity, the tension (i.e. force) acting on the chain at position  $x$  is  $\tau(x) = \rho g x$ .
- (b) Now apply Newton's law: force = mass times acceleration. The tension forces  $\tau(x)$  and  $\tau(x + \Delta x)$  point tangent to the chain. In the horizontal direction, this becomes

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = \tau(x + \Delta x) \sin \beta - \tau(x) \sin \alpha.$$

If we assume (as in the case with a vibrating string) that the chain is close to the equilibrium position compared to the length of the chain, then

$$\sin \alpha \equiv \tan \alpha = \frac{\partial u}{\partial x}(x, t) \quad \sin \beta \equiv \tan \beta = \frac{\partial u}{\partial x}(x + \Delta x, t).$$

Then we have

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = \tau(x + \Delta x, t) \frac{\partial u}{\partial x}(x + \Delta x, t) - \tau(x, t) \frac{\partial u}{\partial x}(x, t).$$

Rearranging this last equation, we find

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{1}{\Delta x} [\tau(x + \Delta x) \frac{\partial u}{\partial x}(x + \Delta x, t) - \tau(x) \frac{\partial u}{\partial x}(x, t)].$$

- (c) Now we take the limit as  $\Delta x \rightarrow 0$ . Then the last equation of part (b) becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} [\tau(x) \frac{\partial u}{\partial x}].$$

Next we substitute  $\tau = \rho g x$  to find

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} [\rho g x \frac{\partial u}{\partial x}] = \rho g [\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2}].$$

Dividing by  $\rho$ , this last equation becomes

$$\frac{\partial^2 u}{\partial t^2} = g [\frac{\partial u}{\partial x} + x \frac{\partial^2 u}{\partial x^2}].$$

It is interesting to notice here that the density  $\rho$  divides out, and does not appear in the final equation! This should not exactly shock you, because gravity tends to act the same on all objects, regardless of their mass (c.f. the famous experiment Galileo did in dropping rocks from the Tower of Pisa).