Selected Solutions for Homework 1

Math 2280

Sept. 26, 2001

1. (Problem 2.1.7)

   We are given functions $f_1, f_2, \ldots$ which are all T-periodic. First we show

   $$f = a_1 f_1 + \cdots a_n f_n$$

   is T-periodic. Indeed,

   $$f(x+T) = a_1 f_1(x+t) + \cdots a_n f_n(x+T) = a_1 f_1(x) + \cdots a_n f_n(x) = f(x).$$

   Now suppose that the series

   $$f(x) = \sum_n a_n f_n(x)$$

   converges for $0 \le x \le T$. Notice that we are *supposing* that the series converges. Then we are to show that f extends to a T-periodic function. This time, for any x choose an integer n such that $x + nT \in [0, T]$ and define

   $$f(x) := \sum_n a_n f(x) = \sum_n a_n f_n(x + nT).$$

   This last sum converges by assumption. That f is periodic is a special case of the last equation (where $n = 1$).

2. (Problem 2.1.8)

   Define $f(x) := \cos(x) + \cos(\pi x)$.

   (a) There is a unique solution to the equation $f(x) = 2$.

   Notice that $x = 0$ is a solution. Now suppose $f(x) = \cos(x) + \cos(\pi x) = 2$. Because cos has a maximal value of 1, this means

   $$\cos(x) = 1 \qquad \cos(\pi x) = 1.$$

   By the properties of cos, this means

   $$x = 2\pi n \qquad x = 2m$$

   for two integers n and m. Thus we must have

   $$2\pi n = 2m \quad \text{or} \quad \pi = \frac{m}{n}.$$

   However, π is an irrational number, so it cannot be written as the ratio of two integers.

   (b) The function f is not periodic.

   Suppose f is periodic with period T. Then

   $$2 = f(0) = f(0+T) = f(T).$$

   However, this is impossible because $x = 0$ is the unique solution to $f(x) = 2$. This does not violate problem 2.1.7 because $\cos(x)$ and $\cos(\pi x)$ have different periods.

3. (Problem 2.1.9c)

   Parts a and b of this problem are very simliar to problem 2.1.7. For part c, we have f, which is T-periodic and g which is not periodic at all. We define

   $$h(x) := g \circ f(x) = g(f(x)).$$

   Then h is T-periodic. Indeed,

   $$h(x+T) = g(f(x+t)) = g(f(x)) = h(x).$$

4. (Problem 2.1.15)

We are given f, which is 2π periodic. and we define

$$F(x) = \int_0^x f(t)dt.$$

(We can choose the lower limit of the integral to be 0 without loss of generality; changing the lower limit amounts to adding a constant to F.)

First we show that $\int_0^{2\pi} f(t)dt = 0$ implies that F is 2π periodic. We wish to show that $F(x) = F(x+2\pi)$. However,

$$\begin{aligned} F(x+2\pi) - F(x) &= \int_0^{x+2\pi} f(t)dt - \int_0^x f(t)dt = \int_0^x f(t)dt + \int_x^{x+2\pi} f(t)dt - \int_0^x f(t)dt \\ &= \int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt = 0. \end{aligned}$$

This proves $F(x+2\pi) - F(x) = 0$, or $F(x+2\pi) = F(x)$.

Next we show that F is 2π-periodic implies $\int_0^{2\pi} f(t)dt = 0$. First observe that $F(0) = \int_0^0 f(t)dt = 0$. Then be periodicity of F, we have

$$0 = F(2\pi) - F(0) = F(2\pi) = \int_0^{2\pi} f(t)dt,$$

which is precisely what we wanted to show.

5. (Problem 2.2.7)

Let $f(x) = |\sin x|$ for $-\pi \le x \le \pi$. I'll compute the Fourier series of f. First notice that f is an even function. Indeed,

$$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x).$$

Therefore, $b_k = 0$ for all $k = 1, 2, 3, \ldots$. Next we compute a_0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x| dx = \frac{1}{\pi}\int_0^{\pi} \sin x dx = \frac{1}{\pi} - \cos x|_0^{\pi} = \frac{-\cos(\pi) + \cos(0)}{\pi} = \frac{2}{\pi}.$$

To compute a_k we will need the following trigonometric identity:

$$\sin x \cos kx = \frac{1}{2}[\sin((k+1)x) - \sin((k-1)x)].$$

Then

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos(kx) dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos kx sx = \frac{1}{\pi}\int_0^{\pi} \sin((k+1)x) - \sin((k-1)x) dx \\ &= \frac{1}{\pi}[-\frac{\cos((k+1)x)}{k+1} + \frac{\cos((k-1)x)}{k-1}]\Big|_0^{\pi}. \end{aligned}$$

This last quantity is 0 if k is odd and $-1/[\pi(k^2-1)]$ if k is even.

6. (Problem 2.2.13)

Let $f(x) = x$ for $-\pi \le x \le \pi$. I'll compute the Fourier series of f. First notice thet f is an odd function, so $a_0 = 0$ and $a_k = 0$ for $k = 1, 2, 3, \ldots$. Now we compute b_k using integration by parts:

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx dx = -\frac{x}{k}\cos kx\Big|_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi} \cos kx dx \\ &= -\frac{1}{k}[\pi\cos(k\pi) + \pi\cos(-k\pi)] = -\frac{2\pi}{k}\cos(k\pi) = 2\pi\frac{(-1)^{k+1}}{k}. \end{aligned}$$

Therefore

$$f(x) = x = 2\pi\sum_1^{\infty} \frac{(-1)^{k+1}}{k}\sin kx.$$

7. (Problem 2.2.18)

   You should have found that the Fourier series for $f(x) = x$ on $[-\pi, \pi]$ is

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

   Now evaluate this series at $x = \pi/2$. Recall that

$$\sin(n\pi/2) = \begin{cases} 0 & n \text{ even} \\ 1 & n = 2k+1 \text{ odd, with } k \text{ even} \\ -1 & \text{otherwise} \end{cases} .$$

   Thus we have

$$\frac{\pi}{2} = 2[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots].$$

   Dividing both sied of the equation by 2, we see

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

8. (Problem 2.2.20)

   We start with the function f, which is the 2π-periodic extension of $f(x) = (1/2)(\pi - x)$ for $0 \le x \le 2\pi$.

   (a) The left and right limits of the derivative of f coincide at $x = 0$.

   First notice that f is smooth away from its break points $2k\pi$, where k is an integer. Indeed, on segments $(2k\pi, 2(k+1)\pi)$, the graph of f is a line segment with slope $-1/2$. Therefore $f'(x) = -1/2$ for $x \neq 2k\pi$. Therefore, the left and right limits of $f'(x)$ as $x \to 0$ are both $-1/2$.

   (b) Is f differentiable at $x = 0$?

   No, f is not even continuous at $x = 0$.

   (c) Let $g(x) = x^{1/3}$. Explain why g is not piecewise smooth at $x = 0$.

   We can compute that away from $x = 0$, $g'(x) = (1/3)x^{-2/3}$, which is not bounded in any neighborhood of $x = 0$. Indeed, $x^{-2/3} \to \infty$ as $x \to 0^+$ and $x^{-2/3} \to -\infty$ as $x \to 0^-$. Since neither limits exist, g cannot be piecewise smooth near $x = 0$.

   (d) Find a 2π periodic function which is not piecewise smooth.

   Following the previous example, we choose $h(x) = \tan(x/2)$.